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ABSOLUTE RIESZ SUMMABILITY OF A FACTORED FOURIER SERIES

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In this paper we discuss the absolute Riesz summability factors for the Fourier series of a function of bounded variation and establish a general theorem. As corollaries to our theorem we get results which are of interest in different directions. While Corollary 1 furnishes a result due to Bosanquet [3] and Corollary 2 extends and improves upon a result of Mohanty and Misra [8], Corollary 3 induces some interesting observations in the theory of absolute Riesz summability.

Let $\lambda=\lambda(\omega)$ be a monotone increasing and differentiable function in (h, ∞) where h is some finite positive number and let it tend to infinity as $\omega\to\infty$. Given a series Σu_n let

$$R_\lambda^k(\omega)=\{\lambda(\omega)\}^{-k}\sum_{n<\omega}\{\lambda(\omega)-\lambda(n)\}^k u_n,\quad k\geq 0.$$

If $R_\lambda^k(\omega)\in BV(h, \infty)$, the series Σu_n is said to be *absolutely summable by the Riesz method of 'order' k and type 'λ'* and we write $\Sigma u_n\in|R, \lambda(\omega), k|$. It is known [6] that the method $|R, \omega, k|$ is equivalent to the Cesàro method $|C, k|$, $k\geq 0$.

Let $f\in L(-\pi, \pi)$ and be a periodic functon with pieriod 2π and let the Fourier series of f be given by

$$f(x)\sim\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)=\Sigma A_n(x).$$

We shall use the following notations:

$$\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\},$$

$$\Phi_\alpha(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-u)^{\alpha-1}\varphi(u)\,du,\quad \alpha>0,$$

$$\Phi_0(t)=\varphi(t),$$

$$\varphi_\alpha(t)=\Gamma(\alpha+1)t^{-\alpha}\Phi_\alpha(t),\quad \alpha\geq 0,$$

$$e(\omega)=\exp(\log\omega)^{1+\beta},\quad \beta\geq 0,$$

$$\xi(\omega, t, s)=\sum_{n<\omega}\{e(\omega)-e(n)\}^{\gamma-1}e(n)\,n^s(\log n)^{-\alpha\beta}\cos nt,$$

$$\xi(\omega,\ t)=\xi(\omega,\ t,\ 0),$$

$$\zeta(\omega,\ t,\ s)=\sum_{n<\omega}\{e(\omega)-e(n)\}^{\gamma-1}e(n)n^{s}(\log n)^{-\alpha\beta}\sin nt,$$

$$g(\omega,\ t)=\frac{1}{\Gamma(1-\alpha)}\int_t^{\pi}(u-t)^{-\alpha}\xi(\omega,\ u)du,\ \ 0<\alpha<1,$$

$$G(\omega,\ t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t u^{\alpha}\frac{\partial}{\partial u}g(\omega,\ u)du,$$

$$Q(\omega,\ s)=\{e(\omega)-e(m)\}^{\gamma-1}e(m)m^{s}(\log m)^{-\alpha\beta},$$

where m is an integer such that $0<\omega-m\leq 1$.

k stands for a suitable constant chosen for convenience in analysis. K, K_1, $K_2\cdots$ denote absolute constants, possibly different at different occurrences.

We prove the following theorem:

THEOREM. *Let* $0<\alpha<1$ *and* $\beta\geq 0$. *Then* $\varphi_{\alpha}(t)\in BV(0,\ \pi)\Rightarrow$

$$\sum_2^{\infty}\frac{A_n(x)}{(\log n)^{\alpha\beta}}\in|R,\ \exp(\log\omega)^{1+\beta},\ \gamma|,\ \gamma>\alpha.$$

To give a neater appearance to the proof of the theorem we work out certain order estimates in the form of a lemma.

LEMMA 1. Let $0<\alpha<\gamma\leq 1$. *Then*

(i) $$\begin{matrix}\xi(\omega,\ t,\ s)\\ \zeta(\omega,\ t,\ s)\end{matrix}=\begin{cases}\omega^{s+1}e^{\gamma}(\omega)(\log\omega)^{-\beta(\alpha+1)}+Q(\omega,\ s),\ s\geq -1,\\ t^{-\gamma}\omega^{s+1-\gamma}e^{\gamma}(\omega)(\log\omega)^{\beta(\gamma-\alpha-1)}+Q(\omega,\ s),\ s>-1;\end{cases}$$

(ii) $$g(\omega,\ t)=\begin{cases}\omega^{\alpha}e^{\gamma}(\omega)(\log\omega)^{-\beta(\alpha+1)}+Q(\omega,\ \alpha-1)\\ t^{-\gamma}\omega^{\alpha-\gamma}e^{\gamma}(\omega)(\log\omega)^{\beta(\gamma-\alpha-1)}+Q(\omega,\ \alpha-1).\end{cases}$$

These estimates are given elsewhere (see [4], Lemma 1) for the case $\gamma=\alpha$. For the sake of completeness we reproduce the modified version of the proof for the present case. We give a proof for the estimates for ξ only, the proof for ζ is similar.

PROOF. $$\xi(\omega,\ t,\ s)\leq\sum_{n<\omega}\{e(\omega)-e(n)\}^{\gamma-1}e(n)n^{s}(\log n)^{-\alpha\beta}$$

$$=\sum_{n\leq\sqrt{\omega}}+\sum_{\sqrt{\omega}<n<\omega}=S_1+S_2,\ \text{say},$$

$$S_1\leq\{e(\omega)-e(\sqrt{\omega})\}^{\gamma-1}e(\sqrt{\omega})(\sqrt{\omega})^{s+1}(\log\sqrt{\omega})^{-\alpha\beta}$$

$$=\{(\omega-\sqrt{\omega})e'(\omega_1)\}^{\gamma-1}e(\sqrt{\omega})(\sqrt{\omega})^{s+1}(\log\sqrt{\omega})^{-\alpha\beta},\ (\sqrt{\omega}<\omega_1<\omega),$$

$$\leq Ke^{\gamma}(\omega)\ \omega^{(s+1)/2}(\log\omega)^{\beta(\gamma-\alpha-1)};$$

$$S_2 \le \int_{\sqrt{\omega}}^{\omega} \{e(\omega)-e(u)\}^{\gamma-1} e(u) u^s (\log u)^{-\alpha\beta} du + Q(\omega, s)$$
$$\le K\omega^{s+1} (\log \sqrt{\omega})^{-\beta(\alpha+1)} \{e(\omega)-e(\sqrt{\omega})\}^{\gamma} + Q(\omega, s)$$
$$\le K\omega^{s+1} (\log \omega)^{-\beta(\alpha+1)} e^{\gamma}(\omega) + Q(\omega, s).$$

This furnishes the first set of estimates.

Let $\omega_1 = \left[\omega - \frac{k}{t}\right]$, and

$$\xi(\omega, t, s) = \left[\sum_{n\le\omega_1} + \sum_{\omega_1+1}^{m}\right] \{e(\omega)-e(n)\}^{\gamma-1} e(n) n^s (\log n)^{-\alpha\beta} \cos nt$$
$$= S_3 + S_4, \text{ say.}$$

As $\{e(\omega)-e(n)\}^{\gamma-1} e(n) n^s (\log n)^{-\alpha\beta}\}$ is ultimately monotone increasing in n for $n<\omega$,

$$S_3 = 0\left[\{e(\omega)-e(\omega_1)\}^{\gamma-1} e(\omega_1)\omega_1^s (\log \omega_1)^{-\alpha\beta} \max_{2\le a<b\le\omega_1} |\textstyle\sum_a^b \cos nt|\right]$$
$$= 0\left[t^{-\gamma} e^{\gamma}(\omega)\omega^{s+1-\gamma} (\log \omega)^{\beta(\gamma-1-\alpha)}\right],$$

and

$$S_4 \le \int_{\omega-\frac{k}{t}}^{\omega} \{e(\omega)-e(u)\}^{\gamma-1} e(u) u^s (\log u)^{-\alpha\beta} du + Q(\omega, s)$$
$$= 0(\omega^{s+1}\left\{e(\omega)-e\left(\omega-\frac{k}{t}\right)\right\}^{\gamma} (\log \omega)^{-\beta(\alpha+1)}) + Q(\omega, s)$$
$$= 0\left[\omega^{s+1}\left\{\left(\frac{k}{t}\right)\frac{e(\omega^*)(1+\beta)(\log \omega^*)^{\beta}}{\omega^*}\right\}^{\gamma} (\log \omega)^{-\beta(\alpha+1)}\right] + Q(\omega, s), \quad \omega_1<\omega^*<\omega,$$
$$= 0\left[\omega^{s+1-\gamma} t^{-\gamma} e^{\gamma}(\omega)(\log \omega)^{\beta(\gamma-\alpha-1)}\right] + Q(\omega, s).$$

(ii) $\Gamma(1-\alpha) g(\omega, t)$

$$= \sum_{n<\omega} \{e(\omega)-e(n)\}^{\gamma-1} e(n)(\log n)^{-\alpha\beta}\left[\int_t^{t+\frac{1}{n}} + \int_{t+\frac{1}{n}}^{\pi}\right](u-t)^{-\alpha} \cos nu\, du$$
$$= \sum_{n<\omega} \{e(\omega)-e(n)\}^{\gamma-1} e(n)(\log n)^{-\alpha\beta}\{\cos n\theta \int_t^{t+\frac{1}{n}} (u-t)^{-\alpha} du +$$
$$+ n^{\alpha}\int_{t+1/n}^{\theta'} \cos nu\, du\} \qquad \text{where } t\le\theta\le t+1/n\le\theta'\le\pi,$$
$$= \sum_{n<\omega} \{e(w)-e(n)\}^{\gamma-1} e(n) n^{\alpha-1} (\log n)^{-\alpha\beta}\left\{\frac{\cos n\theta}{1-\alpha} + \sin n\theta' - \sin(nt+1)\right\}$$
$$= 0\begin{cases} \omega^{\alpha} e^{\gamma}(\omega)(\log \omega)^{-\beta(\alpha+1)} + Q(\omega, \alpha-1), \\ t^{-\gamma}\omega^{\alpha-\gamma} e^{\gamma}(\omega)(\log \omega)^{\beta(\gamma-\alpha-1)} + Q(\omega, \alpha-1), \end{cases}$$

by(i).

PROOF of the theorem

$$\sum_{2}^{\infty} A_n (\log n)^{-\alpha\beta} \in |R\ e(\omega),\ \gamma|,\ \text{iff}$$

$$I=\int_{2}^{\infty} \frac{(\log\ \omega)^{\beta}}{\omega e^{\gamma}(\omega)} \left| \int_{0}^{\pi} \sum_{n<\omega} \{e(\omega)-e(n)\}^{\gamma-1} e(n) (\log\ n)^{-\alpha\beta} \varphi(t) \cos nt\ dt \right| d\omega$$

$$=\int_{2}^{\infty} \frac{(\log\ \omega)^{\beta}}{\omega e^{\gamma}(\omega)} \left| \int_{0}^{\pi} \varphi(t)\xi(\omega,\ t)dt \right| d\omega$$

is convergent.

Now $$\int_{0}^{\pi} \varphi(t)\xi(\omega,\ t)dt = \frac{1}{\Gamma(1-\alpha)} \int_{0}^{\pi} \xi(\omega, u) \int_{0}^{u} (u-t)^{-\alpha} d\Phi_{\alpha}(t)du$$

$$=\frac{1}{\Gamma(1-\alpha)} \int_{0}^{\pi} \int_{u}^{\pi} (u-t)^{-\alpha} \xi(\omega,\ u) d\Phi_{\alpha}(u)$$

$$=\int_{0}^{\pi} g(\omega,\ t) d\Phi_{\alpha}(t)$$

$$=\frac{-1}{\Gamma(\alpha+1)} \int_{0}^{\pi} u^{\alpha} \frac{\partial g}{\partial u}(\omega,\ u)\varphi_{\alpha}(u)du$$

$$=-G(\omega,\pi)\varphi_{\alpha}(\pi)+\int_{0}^{\pi} G(\omega, t)d\Phi_{\alpha}(t).$$

However, taking $\varphi(t)$ to be a constant function in the above, we notice that

$$G(\omega,\pi)=0$$

and therefore

$$\int_{0}^{\pi} \varphi(t)\ \xi\ (\omega,\ t)dt = \int_{0}^{\pi} G(\omega, t)d\varphi_{\alpha}(t).$$

As $\varphi_{\alpha}(t) \in BV(0, \pi)$, to establish the convergence of the integal I it is sufficient to know that

$$J(t)=\int_{2}^{\infty} \frac{(\log\ \omega)^{\beta}}{\omega e^{\gamma}(\omega)} |G(w, t)|d\omega = 0(1),$$

uniformly in t, $0<t<\pi$.

After Lemma 1 we notice that

$$\Gamma(\alpha+1)|G(\omega,\ t)| = |t^{\alpha} g(\omega, t) - \alpha \int_{0}^{t} u^{\alpha-1} g(\omega, u)du|$$

$$\leq K_1 t^{\alpha} \omega^{\alpha} e^{\gamma}(\omega)(\log\ \omega)^{-\beta(\alpha+1)} + K_2\ t^{\alpha} Q(\omega, \alpha-1);$$

and also that

$$\Gamma(\alpha+1)|G(\omega,t)|=|\Gamma(\alpha+1)|G(\omega,\pi)-\int_t^\pi u^\alpha\frac{\partial}{\partial u}g(\omega,u)du|$$

$$=|t^\alpha g(\omega,t)+\alpha\int_t^\pi u^{\alpha-1}g(\omega,u)du|$$

$$\leq K_1t^{\alpha-\gamma}\omega^{\alpha-\gamma}e^\gamma(\omega)(\log\ \omega)^{\beta(\gamma-\alpha-1)}+K_2\ t^\alpha Q(\omega,\ \alpha-1).$$

Therefore

$$J(t)=\Big[\int_2^\tau+\int_\tau^\infty\frac{(\log\omega)^\beta}{\omega e^\gamma(\omega)}|G(\omega,t)|\ d\omega,\ \text{ where } \tau=\frac{k}{t}(\log\frac{k}{t})^\beta,$$

$$\leq K_1t^\alpha\int_2^\tau\omega^{\alpha-1}(\log\ \omega)^{-\alpha\beta}d\omega+K_2t^{\alpha-\gamma}\int_\tau^\infty\frac{(\log\omega)^{\beta(\gamma-\alpha)}}{\omega^{\gamma-\alpha+1}}d\omega$$

$$+K_3t^\alpha\int_2^\infty\frac{(\log\omega)^\beta}{\omega e^\gamma(\omega)}Q(\omega,\alpha-1)\ d\omega$$

$$\leq K_1t^\alpha\tau^\alpha(\log\tau)^{-\alpha\beta}+K_2t^{\alpha-\gamma}\tau^{\alpha-\gamma}(\log\tau)^{\beta(\gamma-\alpha)}$$

$$+K_3\sum_{m=2}^{\omega}\int_m^{m+1}\{e(\omega)-e(m)\}^{\gamma-1}\frac{e(m)(\log\ \omega)^\beta m^{\alpha-1}}{\omega e^\gamma(\omega)(\log\ m)^{\alpha\beta}}d\omega$$

$$\leq K_1t^\alpha\frac{k^\alpha}{t^\alpha}\Big(\log\frac{k}{t}\Big)^{\alpha\beta}\Big\{\log\frac{k}{t}+\beta\log\log\frac{k}{t}\Big\}^{-\alpha\beta}$$

$$+K_2t^{\alpha-\gamma}t^{\gamma-\alpha}\Big(\log\frac{k}{t}\Big)^{(\alpha-\gamma)\beta}\Big\{\log\frac{k}{t}+\beta\log\ \log\frac{k}{t}\Big\}^{\beta(\gamma-\alpha)}$$

$$+K_3\sum_{m=2}^\infty m^{\alpha-1}e^{-\gamma}(m)(\log m)^{-\alpha\beta}\{e(m+1)-e(m)\}^\gamma$$

$$\leq K_1\left\{1+\beta\frac{\log\ \log\frac{k}{t}}{\log\frac{k}{t}}\right\}^{-\alpha\beta}+K_2\left\{1+\beta\frac{\log\ \log\frac{k}{t}}{\log\frac{k}{t}}\right\}^{\beta(\gamma-\alpha)}$$

$$+K_3\sum_2^\infty m^{\alpha-1}e^{-\gamma}(m)(\log\ m)^{-\alpha\beta}\Big[e(m')\frac{(\beta+1)(\log\ m')^\beta}{m'}\Big]^\gamma,\ m<m'<m+1,$$

$$\leq K_1+K_2+K_3\sum_2^\infty\frac{(\log\ m)^{\beta(\gamma-\alpha)}}{m^{1+\gamma-\alpha}}$$

$$\leq K.$$

Corollaries and Remarks

COROLLARY 1. (See Bosanquet [3]). *Let* $0<\alpha<1$. $\varphi_\alpha(t)\in BV(0,\ \pi)\Rightarrow$

$$\Sigma A_n(x) \in |C, \gamma|, \ \gamma > \alpha.$$

COROLLARY 2. *Let* $0<\alpha<1$. $\varphi_\alpha(t) \in BV(0, \pi) \Rightarrow$

$$\sum_2^\infty \frac{A_n(x)}{\log n} \in |R, \ \exp(\log \omega)^{1+1/\alpha}, \ \gamma|, \ \gamma > \alpha.$$

This corollary is an improvement upon the result of Mohanty and Misra [8] who have proved it for $\gamma=1$.

Our theorem though not given for $\alpha=0$, is known to be true when $\beta=0$ alongwith α (Bosanquet [2]). However, for the case $\alpha=0$, we do obtain the following result.

COROLLARY 3. *Let* $\varepsilon>0$ *and* $\beta>\varepsilon$. *Then* $\varphi(t) \in BV(0, \pi) \Rightarrow$

$$\sum_2^\infty \frac{A_n(x)}{(\log n)^\varepsilon} \in |R, \ \exp\ (\log\ \omega)^{1+\beta}, \ \gamma|, \ \textit{for}\ \gamma > \frac{\varepsilon}{\beta}.$$

PROOF. Let $\alpha = \frac{\varepsilon}{\beta}$. Then $\alpha \in (0, 1)$ and the corollary follows after the following lemma:

LEMMA 2. (Bosanquet [1]). *Let* $\delta>0$ *and* $b>a\geq 0$.

Then $\varphi_a(t) \in BV(0, \delta), \Rightarrow \varphi_b(t) \in BV(0, \delta)$.

REMARKS 1. Fix $\beta>0$ in Corollary 3. We note that smaller the $\varepsilon>0$ we choose, better the result we obtain regarding the 'order' γ of the summation a result some what not very common in the theory of summability factors.

2. Similarly, fixing $\varepsilon>0$ in Corollary 3 we see that smaller the γ we choose better the result seems to be arrived at for the summability of the series regarding both the 'order' and the 'type'. In this respect one may note that for intergral k, $|R, \exp(\log \omega)^{1+\beta'}, k| \subset |R, \exp(\log \omega)^{1+\beta}, k|$, $B'>B>-1$(Guha [5], Pati [9]). However, whether this second theorem of consistency, for absoulte Riesz methods also holds for non-intergral 'order' k does not seem to be explicitly known, as the results given to us in this direction are rather of involved nature where it is not easy to verify the conclusioin in a specific case (see Guha [5], Prasad & Pati [10] and also Kuttner [7]).

Finally we have

COROLLARY 4. $\varphi(t) \in BV(0, \pi) \Rightarrow \Sigma \frac{A_n(x)}{(\log n)^\varepsilon} \in |R, \ \exp(\log \omega)^{1+\beta}, \ \gamma|$, *for every*

$\varepsilon>0,\ \gamma>0,\ \beta>-1.$

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