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ADDITIONAL GENERATING RELATIONS FOR CLASSICAL POLYNOMIALS

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1. Introduction

The object of the present paper is to derive four additional Generating relations for the polynomial set $B_n(x,y)$ and its applications thereof. $B_n(x,y)$, the generalization of as many as eighteen classical polynomials such as Laguerre polynomials, Hermite polynomials, Legendre polynomials, Jacobi polynomials, Bedient polynomials etc. has been defined by means of the generating relation

$$(1\cdot1)\quad \sum_{n=0}^{\infty} B_n(x,y)t^n = {}_pF_q[(\alpha_p);\ (\beta_q);\ \nu xt]\ {}_rF_s[(a_r);\ (b_s);\ \mu y^{-m}t^m]$$

valid under the conditions given in [2]. Several other results for the polynomial set $B_n(x,y)$ have also been given in [1] and [2].

The following notations have been used for brevity:

(i) $(m)=1,\ 2,\ \cdots,\ m$

(ii) $(a_p)=a_1,\ a_2,\ \cdots,\ a_p$

(iii) $[(a_p)]_n=(a_1)_n(a_2)_n\cdots(a_p)_n$

(iv) $\Delta(a\ ;\ b)=\frac{b}{a},\ \frac{b+1}{a},\ \cdots,\ \frac{b+a-1}{a}$

(v) $\Delta_k[a\ ;\ b]=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\cdots\left(\frac{b+a-1}{k}\right)$

(vi) $\Delta(m\ ;\ 1+(b_q))=\left(\frac{\gamma+b_j}{m}\right),\ j=1,\ q\ ;\ \gamma=1,\ m$

(vii) $\Delta_k[m\ ;\ 1+(b_q)]=\prod_{j=1}^{q}\prod_{\gamma=1}^{m}\left(\frac{\gamma+b_j}{m}\right)_k$

2. Additional Generating Relations

(i) We have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}B_{n+k}(x,y)v^k t^n}{\underline{|k}\,(c)_n}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(c)_n B_n(x,y)v^k t^{n-k}}{\underline{|k}\,(c)_{n-k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k v^k (1-c-n)_k}{\underline{|k}\, t^k} B_n(x,y)t^n$$

$$=\sum_{n=0}^{\infty}\left(1+\frac{v}{t}\right)^{n+c-1} B_n(x,y)t^n$$

$$=\left(1+\frac{v}{t}\right)^{c-1}\sum_{n=0}^{\infty} B_n(x,y)(v+t)^n$$

$$(2.1)\qquad =\left(1+\frac{v}{t}\right)^{c-1} {}_pF_q[(\alpha_p);(\beta_q);\nu x(v+t)]\,{}_rF_s[(a_r);(b_s);\mu y^{-m}(v+t)^m]$$

Applications

(a) On taking $p=q=0=r=s$; $\nu=2=m$; $y=1$ and $\mu=-1$, we get

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}H_{n+k}(x)\, v^k t^n}{\underline{|n+k}\ \underline{|k}\,(c)_n}=\left(1+\frac{v}{t}\right)^{c-1} e^{2x(v+t)-(v+t)^2}$$

(b) Substituting $p=0=r$; $q=1=s$; $\beta_1=1+\alpha$; $b_1=1+\beta$; $\mu=\frac{1}{2}=\nu$; $y=1=m$ and writing $\frac{x-1}{x+1}$ for x, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+k}(x+1)^{-n-k}\rho_{n+k}^{(\alpha,\beta)}(x)v^k t^n}{(1+\alpha)_{n+k}(1+\beta)_{n+k}\,\underline{|k}\,(c)_n}$$

$$=\left(1+\frac{v}{t}\right)^{c-1} {}_0F_1\left[-;1+\alpha;\frac{1}{2}\ \frac{(x-1)(v+t)}{x+1}\right] {}_0F_1[-;1+\beta;\tfrac{1}{2}(v+t)].$$

These results are believed to be new. Results for other polynomials may be written in a similar way.

(ii) From (1.1), we can write

$$\sum_{n=0}^{\infty}(B_n+B_{n+1})t^n=\sum_{n=0}^{\infty}B_n(x,y)t^n+\frac{1}{t}\sum_{n=0}^{\infty}B_{n+1}(x,y)t^{n+1}$$

$$=\sum_{n=0}^{\infty}B_n(x,y)t^n+\frac{1}{t}\left[\sum_{n=0}^{\infty}B_n(x,y)t^n-B_0(x,y)\right]$$

$$=\left(1+\frac{1}{t}\right)\sum_{n=0}^{\infty}B_n(x,y)\,t^n-\frac{1}{t}B_0(x,y)$$

$$(2.2)\qquad =\left(1+\frac{1}{t}\right){}_pF_q[(\alpha_p);(\beta_q);\nu xt]\,{}_rF_s[(a_r);(b_s);\mu y^{-m}t^m]-\frac{1}{t}$$

as it is easy to verify that $B_0(x,y)=1$.

Applications:

Similar to (2.1), we get the following results, which are believed to be new.

(i) $\sum_{n=0}^{\infty}\left[\frac{H_n(x)}{\underline{|n}}+\frac{H_{n+1}(x)}{\underline{|n+1}}\right]t^n=\left(1+\frac{1}{t}\right)e^{\nu xt-t^2}-\frac{1}{t}.$

(ii) $\sum_{n=0}^{\infty}\left[\frac{L_n^{(\alpha)}(x)}{(1+\alpha)_n}+\frac{L_{n+1}^{(\alpha)}(x)}{(1+\alpha)_{n+1}}\right]=\left(1+\frac{1}{t}\right)e^t\,{}_0F_1[-;\ 1+\alpha;\ -xt]-\frac{1}{t}.$

3. Writing $p=0=q$ in (1.1), we have

$$\sum_{n=0}^{\infty}(c)_n B_{n;\,\nu;-;(b_s)}^{m;\mu;-;(a_r)}(x,y)\,t^n$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{n}{m}]}\frac{(c)_n[(a_r)]_k(\nu x)^{n-mk}\mu^k t^n}{\underline{|k}\,\underline{|n-mk}\,[(b_s)]_k y^{mk}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c)_{n+mk}[(a_r)]_k(\nu x)^n\mu^k t^{n+mk}}{\underline{|k}\,\underline{|n}\ [(b_s)]_k y^{mk}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c+mk)_n(\nu x)^n t^n}{\underline{|n}}\frac{(c)_{mk}[(a_r)]_k\mu^k t^{mk}}{\underline{|k}\ [(b_s)]_k y^{mk}}$$

$$=\sum_{k=0}^{\infty}(1-\nu xt)^{-(c+mk)}\frac{m^{mk}\Delta k[m;\,c]\,[(a_r)]_k\mu^k t^{mk}}{\underline{|k}[(b_s)]_k y^{mk}}$$

$$=(1-\nu xt)^{-c}{}_{m+r}F_s\left[\begin{matrix}\Delta(m;c),\ (a_r)\\ (b_s)\end{matrix};\ \frac{\mu m^m t^m}{y^m(1-\nu xt)^m}\right] \tag{2.3}$$

Applications: On specializing the various parameters similar to (2.1) we arrive at the following results:

(i) $\sum_{n=0}^{\infty}\frac{(c)_n H_n(x)}{\underline{|n}}t^n=(1-2xt)^{-c}{}_2F_0\left[\frac{c}{2},\ \frac{c+1}{2};\ -;\ -\frac{4t^2}{(1-2xt)^2}\right]$

(ii) $\sum_{n=0}^{\infty}\frac{(c)_n L_n^{(\alpha)}(x)t^n}{(1+\alpha)_n}=(1-t)^{-c}{}_1F_1\left[c;\ 1+\alpha;\ \frac{-xt}{1-t}\right].$

4. In an earlier paper [2] we have proved that

$$B_{mm_1+m_2}(x,y)=\sum_{k=0}^{m_1}\frac{[(\alpha_p)]_{m_2+mk}[(a_r)]_{m_1-k}(\nu x)^{m_2+mk}\mu^{m_1-k}}{\underline{|m_2+mk}\,\underline{|m_1-k}[(\beta_q)]_{m_2+mk}[(b_s)]_{m_1-k}y^{m(m_1-k)}}$$

where m_1 and m_2 are positive integers and $m_2<m$.

Substituting $r=0=s$, we have

$$\sum_{m_1=0}^{\infty}(c)_{m_1}B_{mm_1+m_2;\,\nu;(\beta_q);-}^{m;\mu;(\alpha_p);-}(x,y)t^{m_1}$$

$$=\sum_{m_1=0}^{\infty}\sum_{k=0}^{m_1}\frac{(c)_{m_1}(\nu x)^{m_2+mk}\mu^{m_1-k}[(\alpha_p)]_{m_2+mk}t^{m_1}}{\underline{|m_1-k}\,\underline{|m_2+mk}\ y^{m(m_1-k)}[(\beta_q)]_{m_2+mk}}$$

$$=\sum_{m_1=0}^{\infty}\sum_{k=0}^{\infty}\frac{(c+k)_m(\mu t)^{m_1}}{\underline{|m_1}\, m^{mm_1}}\;\frac{(c)_k[(\alpha_p)]_{m_2+mk}(\nu x)^{m_2+mk}t^k}{\underline{|m_2+mk}\,[(\beta_q)]_{m_2+mk}}$$

$$=\sum_{k=0}^{\infty}\left(1-\frac{\mu t}{m}\right)^{-(c+k)}\frac{(c)_k(1)_k[(\alpha_p)]_{m_2}[(\alpha_p)+m_2]_{mk}(\nu x)^{m_2+mk}t^k}{\underline{|k}\,\underline{|m_2}(m_2+1)_{mk}[(\beta_q)]_{m_2}[(\beta_q)+m_2]_{mk}}$$

(2.4)

$$=\frac{(\nu x)^{m_2}[(\alpha_p)]_{m_2}}{\underline{|m_2}\,[(\beta_q)]_{m_2}}\left(1-\frac{\mu t}{m}\right)^{-c}{}_{mp+2}F_{mq+m}\left[\begin{matrix}c,1,\Delta(m;(\alpha_p)+m_2)\\ \Delta(m;m_2+1),\Delta(m;(\beta_q)+m_2)\end{matrix}\;:\;\frac{(\nu x)^m t}{\left(1-\frac{\mu t}{y}\right)}\right].$$

Applications similar to the earlier results can be obtained in this case also.

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