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FIXED POINT THEORY IN SEMI-METRIC SPACE.

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Introduction

Banach's fixed point theorem in a metric space is stated as follows :−i) If (X,ρ) be a complete metric space and T be an operater $T: X\to X$ satisfying the condition $\rho(Tx, Ty)\leq\lambda\rho(x,y)$ for all $x, y\in X$, where $0<\lambda<1$ then there exists a unique fixed point x_0 such that $T(x_0)=x_0$.

By "Semi-metric" on a space X is meant a function ρ on $X\times X$ into R satisfying the following conditions:

(1) $\rho(x, y)=\rho(y, x)\geq 0$

(2) $\rho(x, z)\leq\rho(x, y)+\rho(y, z)$

(3) $\rho(x, x)=0$

where x, y, z are arbitrary points of X. Note that this definition slightly differs from the definition of a "quasi-metric." If in addition to this condition we have $\rho(x, y)=0\Longrightarrow x=y$ then ρ becomes a metric on X.

An equivalent form of Banach's fixed point theorem may be given here as follows:

THEOREM 1. *Let (X, ρ) be a complete semi-metric space and $T: X\to X$ is a continuous mapping such that*

$$\rho(Tx, Ty)\leq\lambda\rho(x, y) \tag{1}$$

for any two points x and $y\in X$, $(0<\lambda<1)$ then there exists at least one point $x\in X$ such that $\rho(Tx, x)=0$. Further, if there be any other point y satisfying

$$\rho(Ty, y)=0 \text{ then } \rho(x, y)=0$$

PROOF. Let $x_n=Tx_{n-1}$ where x_0 is any point of X.

then $\rho(x_{n+k}, x_n)=\rho(T^{n+k}x_0, T^n x_0)\leq\lambda^n\rho(x_k, x_0)$

Now $\rho(x_n, x_0)\leq\rho(x_n, x_{n-1})+\rho(x_{n-1}, x_{n-2})+\cdots\cdots+\rho(x_1, x_0)$

$$\leq\lambda^{n-1}\rho(x_1, x_0)+\cdots\cdots+\rho(x_1, x_0)$$

$\leq(1-\lambda)^{-1}\rho(x_1, x_0)$

$\therefore\ \rho(x_{n+k}, x_n)\leq\lambda^k(1-\lambda)^{-1}\rho(x_1, x_0)$

Now as $0<\lambda<1$, $\{x_n\}$ is Canchy in X.

As X is complete $\{x_n\}$ converges to $x\in X$.

Now $\rho(Tx, x)\leq\rho(Tx, x_{n+1})+\rho(x_{n+1}, x_n)+\rho(x_n, x)$ as $x_{n+1}=Tx_n$, as $n\to\infty$ $x_{n+1}\to Tx$, $x_n\to x$ and $\rho(x_{n+1}, x_n)\to 0$

$\therefore$ for sufficiently large n we have $\rho(Tx, x_{n+1})\leq\epsilon/3$, $\rho(x_{n+1}, x_n)\leq\epsilon/3$, $\rho(x_n, x)\leq\epsilon/3$ so that $\rho(Tx, x)\leq\epsilon$ where ϵ is arbitrary small positive number.

$$\therefore\ \rho(Tx, x)=0 \tag{2}$$

Again, (2) $\Rightarrow T(x)\in\overline{\{x\}}$

As T is continuous. Similarly if $\rho(Ty, y)=0\Rightarrow T(y)\in\overline{\{y\}}$

Now $\rho(x, y)\leq\rho(x, Tx)+\rho(Tx, Ty)+\rho(y, Ty)$

$$\therefore\ \rho(x, y)\leq\rho(Tx, Ty)\leq\lambda\rho(x, y)$$

where $0<\lambda<1\Rightarrow\rho(x, y)=0$

COROLLARY 1. *If X be a complete metric space in theorem 1, then T will have a unique fixed point.*

Theorom 1 may be generralized further as follows:

If (X,ρ) is a complete semi-metric space, and T a continuous mapping of X into itself, such that

$$\rho(T^p x, T^p y)\leq\lambda\rho(x, y) \tag{3}$$

where $0\leq\lambda<1$ and p is a positive integer, then there exists at least one $x\in X$ satisfying the relation $\rho(Tx, x)=0$

Further if $\rho(Ty, y)=0$ for any other $y\in X$

then $\rho(x, y)=0$

THEOREM 2. *Let (X,ρ) be a semi-metric space and T a continuous mapping T: $X\to X$ such that*

$$\rho(Tx, Ty)\leq\lambda\rho(x, y)$$

for any two points x and $y\in X$, $0<\lambda<1$. If $\{T^n x\}$ has a convergent subsequence $\{T^{n_i}x\}$ which converges to x_0, then $\{T^n x\}$ converges to x_0 such that $\rho(x_0, Tx_0)=0$

PROOF. Let $x_n=T^n x$ so that

$\rho(x_1, x_2)\leq\lambda\rho(x_0, x_1)$

$\rho(x_2, x_3) \leq \lambda\rho(x_1, x_2) \leq \lambda^2\rho(x_0, x_1)$

....

$\rho(x_n, x_{n+1}) \leq \lambda^n \rho(x_0, x_1)$

$$\begin{aligned}\rho(x_n, x_{n+p}) &\leq \rho(x_n, x_{n+1}) + \cdots\cdots + \rho(x_{n+p-1}, x_{n+p}) \\ &\leq (\lambda^n + \cdots\cdots + \lambda^{n+p-1})\rho(x_0, x_1) \\ &\leq \lambda^n \frac{1-\lambda^p}{1-\lambda}\rho(x_0, x_1) \qquad (4) \\ &\leq \frac{\lambda^n}{1-\lambda}\rho(x_0, x_1) \to 0 \text{ as } n\to\infty\end{aligned}$$

$\therefore$ $\{x_n\}$ is a Cauchy sequence. So as $\{x_{n_k}\}$ converges to x_0, $\{x_n\}$ also converges to x_0,

$\therefore$ $T^n x \to x_0$

As T is continuous $Tx_0 = \lim_{k\to\infty} T^{n_k+1}x_0$

$$T^2x_0 = \lim_{k\to\infty} T^{n_k+2}x_0$$

$$\begin{aligned}\therefore\ \rho(x_0, Tx_0) &= \lim_{k\to\infty} \rho(T^{n_k}x, T^{n_k+1}x) \\ &= \lim_{k\to\infty} \rho(T^{n_k+1}x, T^{n_k+2}x) \\ &= \rho(Tx_0, T^2x_0) \geq \lambda\rho(x_0, Tx_0)\end{aligned}$$

which is impossible. $\therefore$ $\rho(x_0, Tx_0) = 0$

DEFINITION 1. The mapping T of a semi-metric space X into itself is said to be *contractive* if $\rho(Tx, Ty) < \rho(x, y)$ for $x \neq y \in X$ (5)

DEFINITION 2. Let F denote the family of functions $\alpha(x, y)$ satisfying the following conditions:

$$\left.\begin{aligned}&(1)\ \alpha(x, y) = \alpha(\rho(x, y)) \\ &(2)\ 0 \leq \alpha(\rho) < 1 \quad \text{for each } \rho > 0 \\ &(3)\ \alpha(\rho) \text{ is a monotonically decreasing function of } \rho\end{aligned}\right\} \qquad (6)$$

With the above definition we can write theorem 1 in a more general form as follows

THEOREM 3. *If (X, ρ) is a complete semi-metric space and T is a continuous mapping such that $T: X \to X$ and*

$$\rho(Tx, Ty) \leq \alpha(x, y)\rho(x, y) \qquad (7)$$

for any two points $x, y \in X$, then there exists at least one point ξ, such that $\rho(\xi, T\xi)=0$. Further if there is any other point η such that $\rho(\eta, T\eta)=0$ we shall have $\rho(\xi, \eta)=0$

PROOF. Let $x_n = T^n x_0$, $x_0 \in X$

Now $\rho(x_{n+k}, x_n) = \rho(T^{n+k}x_0, T^n x_0)$

$$\leq \alpha(x_{n+k-1}, x_{n-1}) \cdots\cdots \alpha(x_k, x_0)\rho(x_k, x_0)$$

Now if $\inf_{k,n} \rho(x_{n+k}, x_n) \geq \epsilon$

$$\sup_{n,k} \alpha(x_{n+k}, x_n) \leq \alpha(\epsilon)$$

$$\therefore \rho(x_{n+k}, x_n) \leq [\alpha(\epsilon)]^n \rho(x_k, x_0)$$

$$\rho(x_n, x_0) \leq \rho(x_n, x_{n-1}) + \cdots\cdots + \rho(x_1, x_0)$$

$$\leq \{[\alpha(\epsilon)]^{n-1} + \cdots\cdots + 1\} \rho(x_1, x_0)$$

$$\leq \frac{1-\alpha^n}{1-\alpha} \rho(x_1, x_0),$$

$$\therefore \rho(x_{n+k}, x_n) \leq \frac{\alpha^k}{1-\alpha} \rho(x_1, x_0)$$

Hence $\{x_n\}$ is Cauchy and converges to $x \in X$.

Rest of the argument similar to that of theorem 1.

THEOREM 4. *Let X be a semi-metric space and T a contractive mapping of X into itself such that there exists a point whose sequence of iterates $\{T^n x_0\}$ contains a convergent subsequence $\{T^{n_i} x_0\}$ and if $\xi = \lim_{i \to \infty} T^{n_i} x_0 \in X$ then $\rho(\xi, T\xi)=0$.*

PROOF. Let us suppose $\rho(\xi, T\xi) \neq 0$. Now the sequence $\{T^{n_i+1}x\}$ converges to $T(\xi)$. Let us denote the mapping $r(x, y)$ of $Y = X \times X - \Delta$ (where Δ is the "diagonal" $\{(x, y) | x=y\}$) into the real line, as follows:

$$r(x, y) = \frac{\rho(Tx, Ty)}{\rho(x, y)} \tag{8}$$

This mapping is continuous on Y.

There exists a neighbourhood U of $(\xi, T(\xi)) \in Y$ such that $x, y \in U$ implies

$$0 \leq r(x, y) < R < 1 \tag{9}$$

Now $S_1 = S_1(\xi, \rho)$ and $S_2 = S_2(T(\xi), \rho)$

be open discs centered at ξ and $T(\xi)$ respectively and of radius $\rho > 0$, which is so small that $\rho(\xi, T(\xi)) > 3\rho$. Now due to this assumption

$$\exists x_0 \epsilon X: \ \{T^n(x_0)\} \supset \{T^{n_i}(x_0)\} \text{ with } \lim_{i\to\infty} T^{n_i}(x_0) \in X \tag{10}$$

then exists a positive integer N such that $i>N$, $T^{n_i}(x_0) \in S_1$ and so by $\rho(Tx, Ty) < \rho(x, y)$ we find that $T^{n_i+1}(x_0) \in S_2$ from (10)

$$\rho(\xi, T^{n_i}x_0) + \rho(T^{n_i}x_0, T^{n_i+1}x_0) + \rho(T^{n_i+1}x_0, T(\xi)) > 3\rho$$

$$\therefore \ \rho(T^{n_i}(x_0), T^{n_i+1}(x_0)) > \rho, \ i > N \tag{11}$$

From(8) and (9) we obtain

$$\rho(T^{n_i+1}x_0, T^{n_i+2}x_0) < R\rho(T^{n_i}x_0, T^{n_i+1}x_0)$$

Repeating this argument, when $l>j>N$

$$\rho(T^{n_l}x, T^{n_l+1}x) < \rho(T^{n_{l-1}+1}x, T^{n_{l-1}+2}x)$$

$$< R\rho(T^{n_{l-1}}x, T^{n_{l-1}+1}x) \leq \cdots\cdots < R^{l-j}\rho(T^{n_j}x, T^{n_j+1}x) \to 0, \text{ as } l \to \infty$$

This contradicts (10)

$$\therefore \ \rho(\xi, T\xi) = 0$$

If there is another $\eta \neq \xi$ such that $\rho(\eta, T\eta) = 0$

Now $\rho(\xi, \eta) \leq \rho(\xi, T\xi) + \rho(T\xi, T\eta) + \rho(\eta, T\eta) \leq \rho(T\xi, T\eta) < \rho(\xi, \eta)$

which $\Rightarrow \rho(\xi, \eta) = 0$

THEOREM 5. *If T is a contractive mapping of a metric space X into itself and there exists a subset $M \subset X$ and a point $x_0 \in M$ such that $\rho(x, x_0) - \rho(Tx, Tx_0) \geq 2\rho(x_0, Tx_0)$ for every $x \in X/M$ and T maps M into a compact subset of X, then there exists at least a point ξ such that $\rho(\xi, T\xi) = 0$*

COROLLARY 1. *If T is a contractive mapping such that there exists a point $x_0 \in X$ satisfying $\rho(Tx, Tx_0) \leq \alpha(x, x_0)\rho(x, x_0)$ for every $x \in X$, where $\alpha(x, y) = \alpha(\rho(x, y)) \in F$ and T maps $S(x_0, r)$ with*

$$r = \frac{2\rho(x_0, Tx_0)}{1 - \alpha(2\rho(x_0, Tx_0))}$$

into a compact subset of X, then there exists a point $\xi \in X$ such that $\rho(T\xi, \xi) = 0$.

Instead of proving this theorem we present have a more generalized theorem (No. 7).

THEOREM 6. *If T_1 and T_2 be two continuous mappings of a semi-metric space X into itself such that*

$$\rho(T_1x, T_2y) < \rho(x, y) \ \text{for } x \neq y \in X \tag{12}$$

and then exists some $x_0 \in X$, *the sequence* $\{x_n\}$ *converging to* x *then* $\rho(x, T_1x)=0$. *and* $\rho(x, T_2x)=0$

PROOF. Similar to that of theorem 4,

THEOREM 7. *Let* T_1 *and* T_2 *be two mappings of a semi-metric space* X *into itself satisfying condition* (12) *such that there exists a subset* $M \subset X$ *and a point* $x_0 \in M$ *satisfying* $\rho(x_0, T_ix)-\rho(T_1x_0, T_1T_2x) \geq 2\rho(x_0, Tx_0)$ *for every* $x \in X/M$, $i=1, 2$ *and* $T_1T_2=T_2T_1$ ············(A)

and that T_1 *and* T_2 *map* M *into a compact subset of* X. *Then there exists a point* ξ *such that* $\rho(\xi, T_1\xi)=0=\rho(\xi, T_2\xi)$

PROOF. Let us assume that $x_0 \neq T_1x_0$. The sequence $\{x_n\}$ has the same definition as in theorem 6. Now T_1 and T_2 map M into a compact subset and so to prove this theorem, it will be sufficient to prove that $x_n \in M$ for all n. Rest will follow from theorem 6.

Now $\rho(T_1x, T_2y)<\rho(x, y)$. So

$$\rho(x_{2n}, x_{2n+1})<\rho(x_0, x_1) \text{ and } \rho(x_{2(n+1)}, x_{2n+1})<\rho(x_0, x_1)$$

$$\rho(x_0, x_{2n+1}) \leq \rho(x_0, x_1)+\rho(x_1, x_{2(n+1)})+\rho(x_{2(n+1)}, x_{2n+1})$$

$$=\rho(x_0, T_1x_0)+\rho(T_1x_0, T_2T_1x_{2n})+\rho(x_{2n+1}, x_{2(n+1)})$$

$$\therefore \rho(x_0, T_1x_{2n})-\rho(T_1x_0, T_2T_1x_{2n})<2\rho(x_0, Tx_0)$$

$\therefore$ from(A) it follows that $x_{2n} \in M$.

Similarly

$$\rho(x_0, x_{2(n+1)}) \leq \rho(x_0, T_1x_0)+\rho(T_1x_0, x_{2(n+1)+1})+\rho(x_{2(n+1)+1}, x_{2(n+1)})$$

$$\therefore \rho(x_0, T_2x_{2n+1})-\rho(T_1x_0, T_1T_2x_{2n+1})<2\rho(x_0, T_1x_0)$$

$\therefore$ $x_n \in M$ for every n. Hence the theorem follows.

THEOREM 8. *Let* T_1 *and* T_2 *be two continuous mapping of a complete semi-metric space* X *into itself such that* $\rho(T_1x, T_2y)<\rho(x, y)$ *for* $x \neq y \in X$ *and let there exist a subset* $M \subset X$ *and a point* $x_0 \in M$ *satisfying*

(i) $\rho(x_0, T_1x)-\rho(T_1x_0, T_1T_2x) \geq 2\rho(x_0, T_1x_0)$ *for every* $x \in X/M$ *and* $i=1, 2$

(ii) $\rho(T_1x, T_2y) \leq \alpha(x, y)\rho(x, T_1x)+\beta(x, y)\rho(x, T_2y)$ *for every* $x, y \in M$, *where* $\alpha(x, y)$, $\beta(x, y) \in F$ *and they are decreasing functions of* ρ *such that* $\alpha(\rho(x, y))+\beta(\rho(x, y))<1$

and obviously $\alpha(x, y)=\alpha(y, x)$, $\beta(x, y)=\beta(y, x)$.

Then there exists a point $\xi \in X$ *such that*

$$\rho(T_1\xi, \xi)=0=\rho(T_2\xi, \xi).$$

PROOF. The sequence $\{x_n\}$ is defined as before. Let us suppose that $x_0=T_1x_0$. Now using condition (i), we obtain that $x_n \in M$ for every n. Again $\rho(T_1x, T_2y)$ for $x\neq y$ gives $\rho(x_n, x_{n+1})<\rho(x_0, x_1)$. Now $\{x_n\}$ can be proved to be bounded due to the following measuring:

$$\begin{aligned}\rho(x_0, x_{2n+1}) &\leq \rho(x_0, x_1) + \rho(x_1, x_{2(n+1)})+\rho(x_{2(n+1)}, x_{2n+1})\\ &\leq \rho(x_0, x_1)+\alpha(x_0, x_{2n+1})\rho(x_0, Tx_0)+\beta(x_0, x_{2n+1})\rho(x_{2n+1}, x_{2(n+1)})\\ &\qquad +\rho(x_{2n+1}, x_{2(n+1)})\\ &<2\rho(x_0, T_1x_0)+[\alpha(x_0, x_{2n+1})+\beta(x_0, x_{2n+1})]\rho(x_0, x_1)\end{aligned}$$

For a given $\rho_0>0$, $\rho(x_0, x_{2n+1})\geq\rho_0$, then as $\alpha(\rho)$ $\beta(\rho)$ are monotonic decreasing functions of ρ, so $\rho(x_0, x_{2n+1})\leq[2+\alpha(\rho_0)+\beta(\rho_0)]\rho(x_0, x_1)$

Again,

$$\begin{aligned}\rho(x_0, x_{2(n+1)}) &\leq\rho(x_0, T_1x_0)+\rho(T_1x_0, T_2x_{2n-1})\\ &\qquad +\rho(T_2x_{2n-1}, T_1x_{2n})+\rho(x_{2n+1}, x_{2(n+1)})\\ &\leq\rho(x_0, T_1x_0)+\alpha(x_0, x_{2n-1})\rho(x_0, T_1x_0)+\beta(x_0, x_{2n-1})\rho(x_{2n-1}, x_{2n})\\ &\qquad +\rho(x_{2n}, x_{2n+1})+\rho(x_{2n+1}, x_{2(n+1)})\\ &<3\rho(x_0, x_1)+\rho(x_0, x_1)\alpha(x_0, x_{2n-1})+\rho(x_0, x_1)\beta(x_0, x_{2n-1})\\ &\leq\rho(x_0, x_1)[\beta+\alpha(\rho_0')+\beta(\rho_0')] \text{ for some } \rho_0'>0 \text{ and } \rho(x_0, x_{2n-1})\geq\rho_0'\end{aligned}$$

These show that $\{x_n\}$ is bounded.

Now $\rho(x_1, x_2)\leq\alpha(x_0, x_1)\rho(x_0, T_1x_0)+\beta(x_0, x_1)\rho(x_1, T_2x_1)$

$\therefore$ $\rho(x_1, x_2)\leq\alpha(x_0, x_1)[1-\beta(x_0, x_1)]\rho(x_0, x_1)$

Similarly

$$\rho(x_2, x_3)\leq\frac{\beta(x_1, x_2)}{1-\alpha(x_1, x_2)}\cdot\frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\cdot\rho(x_0, x_1), \text{ so on}$$

In general

$$\rho(x_{2n}, x_{2n+1})\leq\frac{\beta(x_{2n-1}, x_{2n})}{1-\alpha(x_{2n-1}, x_{2n})}\cdot\frac{\alpha(x_{2n-2}, x_{2n-1})}{1-\beta(x_{2n-2}, x_{2n-1})}\cdots\cdots$$

$$\cdots\cdots\frac{\alpha(x_2, x_3)}{1-\beta(x_2, x_3)}\cdot\frac{\beta(x_1, x_2)}{1-\alpha(x_1, x_2)}\cdot\frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\rho(x_0, x_1)$$

$$\rho(x_{2n+1}, x_{2n+2})\leq\frac{\alpha(x_{2n}, x_{2n+1})}{1-\beta(x_{2n+1}, x_{2n})}\cdot\frac{\beta(x_{2n-1}, x_{2n})}{1-\alpha(x_{2n-1}, x_{2n})}\cdots\cdots$$

$$\cdots\cdots\frac{\beta(x_1, x_2)}{1-\alpha(x_1, x_2)}\cdot\frac{\alpha(x_0, x_1)}{1-\beta(x_0, x_1)}\rho(x_0, x_1)$$

Now let $\epsilon>0$, $\rho(x_i, x_{i+1})\geq\epsilon$, $i=0, 1, 2, \cdots\cdots, 2n$

$$\therefore \alpha(x_i, x_{i+1})\leq\alpha(\epsilon), \beta(x_i, x_{i+1})\leq\beta(\epsilon). \ i=0, 1, 2, \cdots\cdots, 2n$$

By our assumption $\alpha(\epsilon)+\beta(\epsilon)<1$

$$\frac{\alpha(\epsilon)}{1-\beta(\epsilon)}<1 \text{ and } \frac{\beta(\epsilon)}{1-\alpha(\epsilon)}<1$$

$$\therefore \rho(x_{2n}, x_{2n+1})\leq r_1^n r_2^n \rho(x_0, x_1)$$

and $\qquad \rho(x_{2n+1}, x_{2(n+1)})\leq r_1^{n+1} r_2^n \rho(x_0, x_1)$

where $\qquad r_1(\epsilon)=\dfrac{\alpha(\epsilon)}{1-\beta(\epsilon)}$ and $r_2(\epsilon)=\dfrac{\beta(\epsilon)}{1-\alpha(\epsilon)}$

$$\therefore \rho(x_{2n}, x_{2n+p})\leq r_1^n r_2^n [1+r_1(1+r_1r_2+r_1^2+r_2^2+\cdots\cdots)$$
$$+r_1r_2(1+r_1r_2+\cdots\cdots)]\rho(x_0, x_1)$$
$$<\frac{r_1^n r_2^n(1+r_1)}{1-r_1r_2}\rho(x_0, x_1)\to 0 \text{ as } n\to\infty$$

Similarly $\qquad \rho(x_{2n+1}, x_{2n+p+1})\to 0$ as $n\to\infty$

$\therefore \{x_n\}$ is a Cauchy sequence.

Now let $\lim\limits_{n\to\infty} \rho(T_2x_{2n-1}, x_{2n+1})=\rho(\lim\limits_{n\to\infty} T_2x_{2n-1}, \lim\limits_{n\to\infty} x_{2n+1})$

$\therefore \rho(T_2\xi, \xi)=0$ [$\because T_2$ is continuous]

$$\lim_{n\to\infty} \rho(T_1x_{2n}, x_{2n+2})=\rho(T_1 \lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} x_{2n+2})\to 0$$

$\therefore \rho(T_1\xi, \xi)=0 \qquad$ [$\because T_1$ is continuous]

This completes the proof.

Putting $T_1=T_2=T$ and $\alpha(x, y)=\beta(x, y)$ on Theorem 8, we obtain,

THEOREM 8*A*. *If T is a continuous mapping of a complete metric space X such that $\rho(Tx, Ty)<\rho(x, y)$ for $x\neq y\in X$ and there exists a subset $M\subset X$ and a point $x_0\in M$ satisfying*

i) *$\rho(x_0, Tx)-(Tx_0, T^2x)\geq 2\rho(x_0, Tx_0)$ for every $x\in X/M$*

ii) *$\rho(Tx, Ty)\leq\alpha(x, y)[\rho(x, Tx)+\rho(y, Ty)]$ for every $x, y\in M$, and $\alpha(x, y)\in F$, then there exists a point ξ, such that $\rho(\xi, T\xi)=0$,*

COROLLARY. *If X is a complete semi-metric space and if $\rho(T_1x_2, T_2y)\leq \alpha(x,y)\ \rho(x, T_1x)+\beta(x, y)\rho(y, T_2y)$ for every x, $y\in X$, then there exists a point ξ such that $\rho(\xi, T_1\xi)=0=\rho(\xi, T_2\xi)$, where α and $\beta\in F$.*

COROLLARY. *If X is complete semi-metric space and α, β are positive constants such that $\alpha+\beta<1$, then if $\rho(T_1x, T_2y) \leq \alpha\rho(x, T_1x)+\beta\rho(y, T_2y)$ then there exists at least a point ξ such that $\rho(\xi, T\xi)=0$. If there is any other point $\alpha+\beta<1$ such that $\rho(\eta, T_i\eta)=0$. $i=1, 2$. then $\rho(\eta, \xi)=0$.*

COROLLARY. *If we make $\alpha=\beta$ in the above corollary, we get the following. If X is a complete semi-metric space and $0<\alpha<1/2$ and $\rho(T_1x, T_2y) \leq \alpha[\rho(x, T_1x)+\rho(y, T_2y)]$* ……(B) *then there exists a point ξ such that $\rho(\xi, T\xi)=0$ and if there is any other $\eta \in X$ satisfying $\rho(\eta, T\eta)=0$, we get $\rho(\xi, \eta)=0$.*

COROLLARY. *Putting $T_1=T_2=T$, the condition* (B) *reduces to*

$$\rho(Tx, Ty) \leq \alpha[\rho(x, Tx)+\rho(y, Ty)]$$

With the help of Theorem 4, we can establish the following theorem:

THEOREM 9. *If T is a contractive mapping of a complete semi-metic space X into itself such that there exists a subset $M \subset X$, and a point $x_0 \in M$ satisfying*

i) $\rho(x, x_0)-\rho(Tx, Tx_0) \geq \rho(x_0, Tx_0)$ *for every* $x \in X/M$

ii) $\rho(Tx, Ty) \leq \lambda(x, y)\rho(x, y)$ *for every* $x, y \in M$, *where* $\lambda(x, y)=\lambda(\rho(x, y))$, $0 \leq \lambda(\rho)<1$

and $\lambda(\rho)$ is a monotonically decreasing function of ρ, then there exists a point ξ such that $\rho(\xi, T\xi)=0$. If there is any $\eta \in X$ such that $\rho(\eta, T\eta)=0$ then $\rho(\xi, \eta)=0$.

COROLLARY. *Taking $M=X$, we get $\rho(Tx, Ty) \leq \lambda(x, y)\rho(x, y)$ for every $x, y \in X$ then there exists a ξ such that $\rho(\xi, T\xi)=0$.*

THEOREM 10. *Let X be a complete semi-metric space and let*

$\rho(T_1x, T_2y) \leq \alpha(x, y)[\rho(x, T_1x)+\rho(y, T_2y)]$ *for every* $x, y \in S(n, r)$,

$S(n, r)$ is an r-neighbourhood of the point x, and if

$$\rho(x, Tx_0)<[1-\lambda(x, Tx)]/r$$

where $\lambda(x, y)=\lambda(\rho(x, y)) \in F$, $\lambda(x, x_1)=\dfrac{\alpha(x, x_1)}{1-\rho(x, x_1)}$

then T_1, T_2 have a point ξ, such that $\rho(\xi, T\xi)=0$.

PROOF. We can prove, as before, $\{x_n\}$ to be a Cauchy sequence. Rest is easy.

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