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A NOTE ON FEEBLY CONTINUOUS FUNCTIONS

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In 1961, Z, Frolik [2] introduced the concept of feebly continuous functions. The definition is as follows: A surjective function $f: X \to Y$ is said to be *feebly continuous* [2] if for every subset B of Y, $\mathrm{Int}_Y(B) \neq \phi$ implies $\mathrm{Int}_X [f^{-1}(B)] \neq \phi$. Recently S.P. Arya and Mamata Deb [1] have dropt the condition "surjective" in this definition and have obtained several properties concerning the restriction and the composition of such functions. The following is the definition due to S. P. Arya and Mamata Deb [1].

DEFINITION A function $f: X \to Y$ is said to be *feebly continuous* [1] if for each open set V of Y, $f^{-1}(V) \neq \phi$ implies $\mathrm{Int}_X [f^{-1}(V)] \neq \phi$.

In the present note we shall deal with feebly continuous functions in the sense of S.P. Arya and Mamata Deb, and shall continue the study of such functions. Throughout this note $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ will denote the closure of A and the interior of A in a topological space X, respectively. Spaces will always mean topological spaces. By $f: X \to Y$ we shall denote a function f of a space X into a space Y.

In [1] , it is shown that the restriction of a feebly continuous function to a dense subset is feebly continuous, but the condition "dense" can not be replaced by "open" or "closed".

THEOREM 1. *Let $f: X \to Y$ be a feebly continuous function. If V is an open set of Y, then the restriction of f to $f^{-1}(V)$ is feebly continuous.*

PROOF By $f_V : f^{-1}(V) \to Y$ we shall denote the restriction of f to $f^{-1}(V)$. Let U be any open set of Y such that $f_V^{-1}(U) \neq \phi$. Then $U \cap V$ is an open set of Y such that $f^{-1}(U \cap V) \neq \phi$. Since f is feebly continuous, we have $\mathrm{Int}_X [f^{-1}(U \cap V)] \neq \phi$ and hence $\mathrm{Int}_X [f_V^{-1}(U)] \neq \phi$. Therefore, we obtain $\mathrm{Int}_{f^{-1}(V)} [f_V^{-1}(U)] \neq \phi$. This implies that f_V is feebly continuous.

THEOREM 2. *Let $f: X \to Y$ and $g: Y \to Z$ be two functions.*

1) *If* $g \circ f$ *is feebly continuous and* g *is open injective, then* f *is feebly continuous.*

2) *If* $g \circ f$ *is feebly continuous and* f *is open surjective, then* g *is feebly continuous.*

PROOF. 1) Let V be any open set of Y such that $f^{-1}(V) \neq \phi$. Then we have $(g \circ f)^{-1}(g(V)) = f^{-1}(V) \neq \phi$ because g is injective. Since g∘f is feebly continuous and g is open, we obtain $\mathrm{Int}_X[f^{-1}(V)] \neq \phi$. This shows that f is feebly continuous.

2) Let W be any open set of Z such that $g^{-1}(W) \neq \phi$. Then we have $(g \circ f)^{-1}(W) \neq \phi$ because f is surjective. Since $g \circ f$ is feebly continuous and f is open, we obtain

$$\phi \neq f(\mathrm{Int}_X[(g \circ f)^{-1}(W)]) \subset \mathrm{Int}_Y[g^{-1}(W)].$$

This shows that g is feebly continuous.

COROLLARY. *Let* $f: X \to Y$ *be an open and feebly continuous surjection. Then, a function* $g: Y \to Z$ *is feebly continuous if and only if* $g \circ f$ *is feebly continuous.*

PROOF. This is an immediate consequence of [1, Theorem 2.6] and Theorem 2.

Let $\{X_\alpha | \alpha \in \mathscr{A}\}$ and $\{Y_\alpha | \alpha \in \mathscr{A}\}$ be two families of spaces with the same set $\mathscr{A}$ of indices. We shall simply denote the product space $\prod\{X_\alpha | \alpha \in \mathscr{A}\}$ and $\prod\{Y_\alpha | \alpha \in \mathscr{A}\}$ by X and Y, respectively. Similarly to the case of continuous functions, feebly continuous functions have the following property.

THEOREM 3. *Let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a function for each* $\alpha \in \mathscr{A}$ *and* $f: X \to Y$ *a function defined by* $f((x_\alpha)) = (f_\alpha(x_\alpha))$ *for each point* $(x_\alpha) \in X$. *Then,* f *is feebly continuous if and only if* f_α *is feebly continuous for each* $\alpha \in \mathscr{A}$.

PROOF. Necessity. Suppose that f is feebly continuous. Let α be an arbitrarily fixed index of $\mathscr{A}$. Let $p_\alpha : X \to X_\alpha$ and $q_\alpha : Y \to Y_\alpha$ be the projections. Since f is feebly continuous and q_α is continuous, $q_\alpha \circ f$ is feebly continuous [1, Theorem 2.7]. By the definition of f, we have $f_\alpha \circ p_\alpha = q_\alpha \circ f$. Therefore, $f_\alpha \circ p_\alpha$ is feebly continuous. Since p_α is open and surjective, by Theorem 2, f_α is feebly continuous.

Sufficiency. Suppose that f_α is feebly continuous for each $\alpha \in \mathscr{A}$. Let V be any basic open set in Y such that $f^{-1}(V) \neq \phi$. Then, we can denote

$$V=\prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\alpha\neq\alpha_j} Y_\alpha ,$$

where V_{α_j} is an open set of Y_{α_j} for each j $(1\leqq j\leqq n)$. It follows from $f^{-1}(V)\neq\phi$ that $f_{\alpha_j}^{-1}(V_{\alpha_j})\neq\phi$ for each j $(1\leqq j\leqq n)$. Since f_α is feebly continuous for each $\alpha\in\mathscr{A}$, we obtain $\mathrm{Int}_{X_{\alpha_j}}[f_{\alpha_j}^{-1}(V_{\alpha_j})]\neq\phi$ for each j $(1\leqq j\leqq n)$. Therefore, by using (3) of [3, p. 152], we have

$$\mathrm{Int}_X[f^{-1}(V)]=\prod_{j=1}^{n}\mathrm{Int}_{X_{\alpha_j}}[f_{\alpha_j}^{-1}(V_{\alpha_j})]\times\prod_{\alpha\neq\alpha_j} X_\alpha\neq\phi.$$

Now, let W be any open set in Y such that $f^{-1}(W)\neq\phi$. Then there exist a point $x\in f^{-1}(W)$ and a basic open set V_X in Y such that $f(x)\in V_X\subset W$. Since $f^{-1}(V_X)\neq\phi$, we have $\mathrm{Int}_X[f^{-1}(V_X)]\neq\phi$ and hence $\mathrm{Int}_X[f^{-1}(W)]\neq\phi$.

This shows that f is feebly continuous.

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REFERENCES

[1] S.P. Arya and Mamata Deb, *On mappings almost continuous in the sense of Frolik*, Math. Student 41(1973), 311—321.

[2] Z. Frolik, *Remarks concerning the invariance of Baire spaces under mappings*, Czechoslovak. Math. J. 11(86) (1961), 381—385.

[3] K. Kuratowski, *Topology, Vol.* 1 (English transl.), Academic Press, New York and London, 1966.