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$T_{\frac{1}{2}}$-SPACES

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1. Introduction

Levine [4] defines a subset of a topological space to be generalized closed (g-closed) if its closure is contained in each of its neighborhoods, and he shows that g-closed sets possess many of the familiar and important properties of closed sets. Of some interest, then, are the $T_{\frac{1}{2}}$-spaces — the spaces in which the closed sets and the g-closed sets coincide. This paper examines such spaces, furnishing characterizations independent of the notion of g-closed sets; investigating their behavior with respect to subspaces, transformations, and products; and providing structure theorems for minimal and maximal $T_{\frac{1}{2}}$ topologies on a given set.

2. Definitions and characterizations

DEFINITION 2.1. (Levine, [4]) In a topological space X, $A \subset X$ is *g-closed* if $c(A) \subset O$ when $A \subset O$ and O is open, where "c" denotes the closure operator.

DEFINITION 2.2. (Levine, [4]) X is a $T_{\frac{1}{2}}$*-space* iff every g-closed subset of X is closed.

DEFINITION 2.3. (Thron, [5]) X is a T_D*-space* iff the derived set of each singleton is closed.

DEFINITION 2.4. X is a *door space* iff each subset of X is either open or closed. (See Kelley, [3])

THEOREM 2.5. *X is $T_{\frac{1}{2}}$ iff for each $x \in X$, either $\{x\}$ is open or $\{x\}$ is closed.*

PROOF. Necessity: Suppose X is $T_{\frac{1}{2}}$ and for some $x \in X$, $\{x\}$ is not closed. Since X is the only neighborhood of $\mathscr{C}\{x\}$ ("$\mathscr{C}$" denotes the complement operator), $\mathscr{C}\{x\}$ is g-closed and thus closed. Hence $\{x\}$ is open.

Sufficiency: Let $A \subset X$ be *g-closed* with $x \in c(A)$. If $\{x\}$ is open, $\phi \neq \{x\} \cap A$. Otherwise $\{x\}$ is closed and $\phi \neq c(x) \cap A = \{x\} \cap A$ by Levine [4], Theorem 2.2. In either case $x \in A$ and so A is closed.

COROLLARY 2.6. *X is $T_{\frac{1}{2}}$ iff every subset of X is the intersection of all open sets and all closed sets containing it.*

PROOF. Necessity: Let X be $T_{\frac{1}{2}}$ with $B \subset X$ arbitrary. Then $B = \bigcap \{\mathscr{C}\{x\}: x \notin B\}$, an intersection of open and closed sets by Theorem 2.5. The result follows.

Sufficiency: For each $x \in X$, $\mathscr{C}\{x\}$ is the intersection of all open sets and all closed sets containing it. Thus $\mathscr{C}\{x\}$ is either open or closed and X is $T_{\frac{1}{2}}$.

COROLLARY 2.7. (*a*) [4, *Theorem* 5.3]: *A T_1-space is $T_{\frac{1}{2}}$*

(*b*) *A door space is $T_{\frac{1}{2}}$.*

PROOF. Both results follow form Theorem 2.5.

EXAMPLE 2.8. Neither implication in the previous corollary is reversible. For if $X = \{a, b, c, d\}$ and $\mathscr{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$, then $(X, \mathscr{T})$ is a $T_{\frac{1}{2}}$-space by Theorem 2.5. But $(X, \mathscr{T})$ is not T_1 since $\{a\}$ is not closed, nor is it a door space, since $\{a, c\}$ is neither open nor closed.

THEOREM 2.9. *If X is $T_{\frac{1}{2}}$, then X is T_D (and thus T_0).*

PROOF. For $x \in X$, $\{x\}$ is either open or closed. If $\{x\}$ is open, $\{x\}' = c(x) \setminus \{x\}$ is closed, while if $\{x\}$ is closed, $\{x\}' = \phi$.

EXAMPLE 2.10. *A* T_D-space need not be $T_{\frac{1}{2}}$. For, if $X = \{a, b, c\}$ and $\mathscr{T} = \{\phi, \{a\}, \{a, b\}, X\}$, then $(X, \mathscr{T})$ is not $T_{\frac{1}{2}}$ since $\{b\}$ is neither open nor closed. But $(X, \mathscr{T})$ is T_D since $\{a\}' = \{b, c\}$, $\{b\}' = \{c\}$, and $\{c\}' = \phi$, all of which are closed.

3. Subspaces and transformations

THEOREM 3.1. *If X is $T_{\frac{1}{2}}$ with $Y \subset X$, then Y is $T_{\frac{1}{2}}$.*

PROOF. For $y \in Y$, $\{y\}$ is either X-open or X-closed, and thus $\{y\}$ is either Y-open or Y-closed.

EXAMPLE 3.2. Before considering conditions under which the image of a $T_{\frac{1}{2}}$- space is $T_{\frac{1}{2}}$, we introduce the following example: Let $X = \{1, 2, 3, \cdots\}$ be the natural numbers with topology $\mathscr{T} = \{\phi, \{1\}\} \cup \{U: 1 \in U \text{ and } \mathscr{C}U \text{ is finite}\}$, and let $Y = \{a, b, c\}$ with topology $\mathscr{V} = \{\phi, \{a\}, Y\}$. Define $f: X \to Y$ by

$$f(1) = a$$

$$f(2n) = b \text{ for } n = 1, 2, \cdots$$

$$f(2n+1)=c \text{ for } n=1,2,\cdots$$

Then f is continuous, open, and onto. But by Theorem 2.5, $(X,\mathscr{T})$ is $T_{\frac{1}{2}}$ while $(Y,\mathscr{V})$ is not.

THEOREM 3.3. *If X is $T_{\frac{1}{2}}$* and *$f:X\to Y$ is continuous, closed, and onto, then Y is $T_{\frac{1}{2}}$.*

PROOF. Let $B\subset Y$ be g-closed. By Levine [4], Theorem 6.3, $f^{-1}[B]$ is g-closed and thus closed in X. Hence $B=f[f^{-1}[B]]$ is closed in Y, and Y is $T_{\frac{1}{2}}$.

THEOREM 3.4. *Let $(X,\mathscr{T})$ be $T_{\frac{1}{2}}$ and let $f: X\to Y$ be an open, onto map (not necessarily continuous) such that for each $y\in Y$, $f^{-1}[\{y\}]$ is a finite set. Then $(Y,\mathscr{U})$ is $T_{\frac{1}{2}}$.*

PROOF. We shall use Theorem 2.5. Let $y\in Y$. By hypothesis, $f^{-1}[\{y\}]=\{x_1, x_2, \cdots, x_n\}$. If, for some i, $\{x_i\}\in\mathscr{T}$ then $\{y\}=\{f(x_i)\}\in\mathscr{U}$ since f is open. Otherwise, $\mathscr{C}\{x_i\}\in\mathscr{T}$ for all $i=1, 2, \cdots n$ and thus $\mathscr{C}\{y\}=f[\mathscr{C}\{x_1\}\cap\cdots\cap\mathscr{C}\{x_n\}]\in\mathscr{U}$. It follows that $(Y,\mathscr{U})$ is $T_{\frac{1}{2}}$.

COROLLARY 3.5. *The homeomorphic image of a $T_{\frac{1}{2}}$-space is $T_{\frac{1}{2}}$.*

4. Products

THEOREM 4.1. *Let $X=\times\{X_\alpha: \alpha\in\Delta\}$. Then if X is $T_{\frac{1}{2}}$, X_α is $T_{\frac{1}{2}}$ for all $\alpha\in\Delta$.*

PROOF. X contains a subspace homeomorphic to X_α. Use Theorem 3.1 and Corollary 3.5.

REMARK 4.2. In contrast to the T_0, T_1, and T_2 separation axioms, the converse of Theorem 4.1 is false. See Levine [4], Example 7.4. In order to derive necessary and sufficient conditions under which a product space is $T_{\frac{1}{2}}$, we distinguish two cases — when the product is infinite (that is, when there are an infinite number of non-singleton factors) and when the product is finite. We begin with a simple lemma:

LEMMA 4.3 *Let $X=\times\{X_\alpha: \alpha\in\Delta\}$ where Δ is infinite. Then X is $T_{\frac{1}{2}}$ iff X is T_1.*

PROOF. The sufficiency is Corollary 2.7 (a). To prove necessity, let $x\in X$ and note that $\{x\}$ is not open in the product topology since there are infinitely many non-singleton factors in X. By Theorem 2.5, $\{x\}$ is closed and X is T_1.

THEOREM 4.4. *Let $X=\times\{X_\alpha: \alpha\in\Delta\}$ where Δ is infinite. Then X is $T_{\frac{1}{2}}$ iff X_α is T_1 for all $\alpha\in\Delta$.*

PROOF. Apply the previous lemma and the fact that a product space is T_1 iff each factor is T_1.

REMARK 4.5. Theorem 4.4 shows that the distinction between $T_{\frac{1}{2}}$ and T_1 vanishes in infinite product spaces. A different situation exists in the case of finite products, where we can relax the T_1 condition on one of the factors if we put severe restrictions upon the others:

THEOREM 4.6. *Let $(X, \mathscr{T})=\times\{(X_i, \mathscr{T}_i): i=1, 2, \cdots, n\}$. Then $(X, \mathscr{T})$ is $T_{\frac{1}{2}}$ iff one of the following conditions holds:*

(a) *$(X_i, \mathscr{T}_i)$ is T_1 for all i.*

or

(b) *For some k, $(X_k, \mathscr{T}_k)$ is $T_{\frac{1}{2}}$ but not T_1, while $(X_i, \mathscr{T}_i)$ is discrete for all $i\neq k$.*

PROOF. Necessity: Suppose $(X, \mathscr{T})$ is $T_{\frac{1}{2}}$ and (a) does not hold. Then for some k, $(X_k, \mathscr{T}_k)$ is not T_1, although $(X_k, \mathscr{T}_k)$ is $T_{\frac{1}{2}}$ by Theorem 4.1. Fix $i\neq k$. We assert $(X_i, \mathscr{T}_i)$ is discrete. For otherwise, there is an $x_i\in X_i$ such that $\{x_i\}\notin\mathscr{T}_i$. Moreover, for some $x_k\in X_k$, $\{x_k\}$ is not $\mathscr{T}_k$-closed. Define $x^*\in X$ by

$$x^*(k)=x_k$$
$$x^*(i)=x_i$$
$$x^*(j)\in X_j \text{ arbitrary for } j\neq k, i.$$

If $\{x^*\}\in\mathscr{T}$, then $P_i[\{x^*\}]=\{x_i\}\in\mathscr{T}_i$, a contradiction; and if $\{x^*\}$ is $\mathscr{T}$-closed, then $\{x_k\}$ is $\mathscr{T}_k$-closed, again a contradiction. By Theorem 2.5 we conclude $(X_i, \mathscr{T}_i)$ is discrete.

Sufficiency: If (a) holds, $(X, \mathscr{T})$ is T_1 and thus $T_{\frac{1}{2}}$. If (b) holds, then for some k, $(X_k, \mathscr{T}_k)$ is $T_{\frac{1}{2}}$ but not T_1, while $(X_i, \mathscr{T}_i)$ is discrete for $i\neq k$. Let $x\in X$. If $\{x(k)\}\in\mathscr{T}_k$, then $\{x\}=\{\times\{x(j): 1\leq j\leq n\}\}\in\mathscr{T}$. Otherwise $\{x(k)\}$ is

$\mathscr{T}_\beta$-closed and thus $\{x\}$ is $\mathscr{T}$-closed. Hence $(X, \mathscr{T})$ is $T_{\frac{1}{2}}$.

5. The $T_{\frac{1}{2}}$ property and the lattice of topologies

THEOREM 5.1. *If $(X, \mathscr{T})$ is $T_{\frac{1}{2}}$ and $\mathscr{T}\subset\mathscr{U}$*, then $(X, \mathscr{U})$ is $T_{\frac{1}{2}}$.

PROOF. For $x\in X$, either $\{x\}\in\mathscr{T}\subset\mathscr{U}$ or $\mathscr{C}\{x\}\in\mathscr{T}\subset\mathscr{U}$.

EXAMPLE 5.2. The $T_{\frac{1}{2}}$ property is not transferred to coarser topologies nor even to infima. For, if $X=\{a, b\}$ with $\mathscr{T}=\{\phi, \{a\}, X\}$ and $\mathscr{U}=\{\phi, \{b\}, X\}$, then $(X, \mathscr{T})$ and $(X, \mathscr{U})$ are $T_{\frac{1}{2}}$ while $(X, \mathscr{T}\cap\mathscr{U})$ is not. However, we can prove:

THEOREM 5.3. *If $(X_\alpha, \mathscr{T}_\alpha)$ is $T_{\frac{1}{2}}$ for all $\alpha\in\Delta$, and if $\{\mathscr{T}_\alpha: \alpha\in\Delta\}$ is a totally ordered family with respect to inclusion, then $(X, \cap\{\mathscr{T}_\alpha: \alpha\in\Delta\})$* is $T_{\frac{1}{2}}$.

PROOF. Let $x\in X$ and suppose $\{x\}\notin\cap\{\mathscr{T}_\alpha: \alpha\in\Delta\}$. Then $\{x\}\notin\mathscr{T}_\beta$ for some $\beta\in\Delta$ and so $\mathscr{C}\{x\}\in\mathscr{T}_\beta$. We assert that $\mathscr{C}\{x\}\in\mathscr{T}_\alpha$ for all $\alpha\in\Delta$. For if $\alpha\in\Delta$ and $\mathscr{T}_\beta\subset\mathscr{T}_\alpha$, then $\mathscr{C}\{x\}\in\mathscr{T}_\alpha$. Otherwise, by total ordering, $\mathscr{T}_\alpha\subset\mathscr{T}_\beta$ and if $\mathscr{C}\{x\}\notin\mathscr{T}_\alpha$ then $\{x\}\in\mathscr{T}_\alpha\subset\mathscr{T}_\beta$, a contradiction. Thus $\mathscr{C}\{x\}\in\cap\{\mathscr{T}_\alpha: \alpha\in\Delta\}$ and so $(X, \cap\{\mathscr{T}_\alpha: \alpha\in\Delta\})$ is $T_{\frac{1}{2}}$.

COROLLARY 5.4. *For $\mathscr{T}$ any topology on X, there is a topology $\mathscr{U}$ on X such that*:

(a) $\mathscr{T}\subset\mathscr{U}$

(b) $(X, \mathscr{U})$ *is* $T_{\frac{1}{2}}$.

and

(c) *If $(X, \mathscr{V})$ is $T_{\frac{1}{2}}$ for $\mathscr{T}\subset\mathscr{V}\subset\mathscr{U}$, then $\mathscr{V}=\mathscr{U}$.*

PROOF. Let $\mathfrak{A}=\{\mathscr{T}_\alpha: \alpha\in\Delta\}$ be the indexed family of all $T_{\frac{1}{2}}$ topologies on X finer than $\mathscr{T}$. We note that $\mathfrak{A}\neq\phi$ since the discrete topology is $T_{\frac{1}{2}}$. Moreover, if $\{\mathscr{T}_\alpha: \alpha\in\Delta^*\}$ is a subset of $\mathfrak{A}$ totally ordered with respect to inclusion, then $\mathscr{T}^*=\cap\{\mathscr{T}_\alpha: \alpha\in\Delta^*\}$ is $T_{\frac{1}{2}}$ with $\mathscr{T}\subset\mathscr{T}^*$. Thus $\mathscr{T}^*\in\mathfrak{A}$ and by Zorn's Lemma, $\mathfrak{A}$ contains a minimal element $\mathscr{U}$ which satisfies properties (a)-(c) above.

6. Minimal $T_{\frac{1}{2}}$ topologies

REMARK 6.1. Letting $\mathscr{T}$ be the indiscrete topology in Corollary 5.4, we see that on any set X there is at least one topology minimal with respect to the property of being $T_{\frac{1}{2}}$.

We shall determine the structure of such topologies, although the cases where X is infinite and X is finite must be treated separately. Some lemmas are necessary:

LEMMA 6.2. *Suppose X contains more than one point and $\mathcal{T}$ is the discrete topology on X. Then $\mathcal{T}$ is not a minimal $T_{\frac{1}{2}}$ topology on X.*

PROOF. Fix $x \neq y$ in X and define the $T_{\frac{1}{2}}$ topology $\mathcal{U} = \{U: U = \phi \text{ or } x \in U\} \subsetneq \mathcal{T}$.

LEMMA 6.3. *Let X be finite with $(X, \mathcal{U})$ $T_{\frac{1}{2}}$. Suppose there is a $c \in X$ such that $\{c\}$ is closed and $\{x \in X: \{x\} \in \mathcal{U}\} \subset \{c\}$. Then $\mathcal{U}$ is discrete.*

PROOF. $\{c\}$ is closed, and for $x \neq c$, $\{x\} \notin \mathcal{U}$ and thus $\{x\}$ is closed by Theorem 2.5. It follows that $(X, \mathcal{U})$ is T_1 and thus discrete.

LEMMA 6.4. *Let $X \neq \phi$ with $A \subset X$ and define $\mathcal{U} = \{U: U \subset A, \text{ or } A \subset U \text{ and } \mathcal{C}U$ is finite$\}$. Then $\mathcal{U}$ is a $T_{\frac{1}{2}}$ topology on X.*

PROOF. Apply Theorem 2.5.

LEMMA 6.5. *Suppose $(X, \mathcal{T})$ is a minimal $T_{\frac{1}{2}}$-space where X contains more than one point. Define*

$$A = \{x: \{x\} \in \mathcal{T} \text{ and } \mathcal{C}\{x\} \notin \mathcal{T}\}$$
$$B = \{x: \{x\} \notin \mathcal{T} \text{ and } \mathcal{C}\{x\} \in \mathcal{T}\}$$
$$C = \{x: \{x\} \in \mathcal{T} \text{ and } \mathcal{C}\{x\} \in \mathcal{T}\}$$

Then:

(a) $X = A \cup B \cup C$

(b) $B \neq \phi$

and

(c) $C = \phi$

PROOF.

(a) This is a restatement of Theorem 2.5.

(b) If $B = \phi$, $\mathcal{T}$ is discrete, contradicting Lemma 6.2.

(c) Suppose $c \in C$ and let $A^* = (A \cup C) \setminus \{c\}$.

Defining $\mathcal{U} = \{U: U \subset A^*, \text{ or } A^* \subset U \text{ and } \mathcal{C}U \text{ is finite}\}$, we conclude from Lemma 6.4 that $(X, \mathcal{U})$ is $T_{\frac{1}{2}}$ and assert that $\mathcal{U} \subset \mathcal{T}$. For, if $U \in \mathcal{U}$ and $U \subset A^*$, then $U = \bigcup \{\{x\}: x \in A^* \cap U\} \in \mathcal{T}$. Alternately, if $U \not\subset A^*$, then $A^* \subset U$ with $\mathcal{C}U = \{x_1, \cdots,$

$x_n\}$. But for each i, $x_i \notin A^*$ and thus either $x_i = c$ or $\{x_i\} \notin \mathscr{T}$. In either case, $\{x_i\}$ is $\mathscr{T}$-closed and so $U = \bigcap \{\complement\{x_i\}: 1 \leq i \leq n\} \in \mathscr{T}$. Hence $\mathscr{U} \subset \mathscr{T}$ and, by minimality, $\mathscr{U} = \mathscr{T}$. Since $\{c\} \in \mathscr{T} = \mathscr{U}$, either $\{c\} \subset A^*$ or $A^* \subset \{c\}$ with $\complement\{c\}$ finite. The first possibility is dismissed by the definition of A^*. In the second case, we conclude X is finite and $A^* = \phi$. Thus $\{x: \{x\} \in \mathscr{U}\} = \{x: \{x\} \in \mathscr{T}\} = A \cup C \subset \{c\}$. By Lemma 6.3, $\mathscr{U} = \mathscr{T}$ is discrete, contradicting Lemma 6.2. We thus reject the original hypothesis that $C \neq \phi$.

THEOREM 6.6. *Suppose X is an infinite set. Then $\mathscr{T}$ is a minimal $T_{\frac{1}{2}}$ topology on X iff there is an $A \subsetneq X$ such that $\mathscr{T} = \{O: O \subset A$, or $A \subset O$ and $\complement O$ is finite$\}$.*

PROOF. Necessity: Suppose $\mathscr{T}$ is minimal $T_{\frac{1}{2}}$ and define A and B as in Lemma 6.5. Then $X = A \cup B$ with $A \subsetneq X$. Define $\mathscr{U} = \{O: O \subset A$, or $A \subset O$ and $\complement O$ is finite$\}$. Then $(X, \mathscr{U})$ is $T_{\frac{1}{2}}$ by Lemma 6.4 and we need only show $\mathscr{T} = \mathscr{U}$. But, if $O \in \mathscr{U}$, either $O \subset A$ or $A \subset O$ with $\complement O$ finite. In the first case, $O \in \mathscr{T}$ clearly. Otherwise, $\complement O = \{x_1, \cdots, x_n\}$ with $x_i \in B$ for all i, proving $O = \bigcap \{\complement\{x_i\}: 1 \leq i \leq n\} \in \mathscr{T}$. Hence $\mathscr{U} \subset \mathscr{T}$, and by minimality it follows that $\mathscr{U} = \mathscr{T}$.

Sufficiency: If $\mathscr{T} = \{O: O \subset A$, or $A \subset O$ and $\complement O$ is finite$\}$ for some $A \subsetneq X$, then $(X, \mathscr{T})$ is $T_{\frac{1}{2}}$ and we must show minimality. Suppose $(X, \mathscr{U})$ is $T_{\frac{1}{2}}$ with $\mathscr{U} \subset \mathscr{T}$. Define $A^* \subset X$ by $A^* = \{x: \{x\} \in \mathscr{U}\}$. We assert that $A = A^*$. For, if $x \in A^*$, $\{x\} \in \mathscr{U} \subset \mathscr{T}$ and thus either $\{x\} \subset A$ or $A \subset \{x\}$ with $\complement\{x\}$ finite. The latter possibility is dismissed since X is infinite. Thus $x \in A$ and $A^* \subset A$. Conversely, suppose $x \in A$ but $x \notin A^*$. Then $\{x\} \notin \mathscr{U}$ and so $\complement\{x\} \in \mathscr{U} \subset \mathscr{T}$. Consequently, either $\complement\{x\} \subset A$ or $A \subset \complement\{x\}$. In the first case, $X = \{x\} \cup \complement\{x\} \subset A \subsetneq X$, while in the second case, $x \in A \subset \complement\{x\}$ and both are contradictions. We conclude $A = A^*$. But now, for $O \in \mathscr{T}$, if $O \subset A$, then $O \subset A^*$ implies $O \in \mathscr{U}$. Otherwise, $A \subset O$ with $\complement O = \{x_1, x_2, \cdots, x_n\}$. Then for each i, $\{x_i\} \notin A = A^*$ and so $\complement\{x_i\} \in \mathscr{U}$, implying $O \in \mathscr{U}$. Thus $\mathscr{T} \subset \mathscr{U}$ and it follows that $\mathscr{T}$ is a minimal $T_{\frac{1}{2}}$ topology.

REMARK 6.7. The previous result shows that the minimal $T_{\frac{1}{2}}$ topologies are composed of some "very small" open sets (the subsets of A) and some "very large" ones (supersets of A with finite complements). A similar result for finite X requires only a minor modification:

THEOREM 6.8. *Suppose X is a finite set containing more than one point. Then*

$\mathscr{T}$ is a minimal $T_{\frac{1}{2}}$ topology on X iff there is an $\phi \neq A \subsetneq X$ such that $\mathscr{T} = \{O$: $O \subset A$ or $A \subset O\}$.

PROOF. Necessity: Again define A and B as in Lemma 6.5 and note that if $A = \phi$, $\mathscr{T}$ is cofinite and thus discrete, a contradiction. The remainder of the necessary condition follows exactly as in Theorem 6.6.

Sufficiency: Suppose that for some $\phi \neq A \subsetneq X$, $\mathscr{T} = \{O : O \subset A$ or $A \subset O\}$. As in the previous theorem, let $(X, \mathscr{U})$ be $T_{\frac{1}{2}}$ with $\mathscr{U} \subset \mathscr{T}$ and define $A^* = \{x: \{x\} \in \mathscr{U}\}$. We assert $A^* \subset A$. For if $x \in A^*$, $\{x\} \in \mathscr{U} \subset \mathscr{T}$ and thus either $\{x\} \subset A$ or $\phi \neq A \subset \{x\}$. In either case, $x \in A$ and so $A^* \subset A$. We now prove $A \subset A^*$ and the minimality of $\mathscr{T}$ exactly as in Theorem 6.6.

COROLLARY 6.9. *If $\mathscr{T}$ is a minimal $T_{\frac{1}{2}}$ topology on X, then $(X, \mathscr{T})$ is compact and connected.*

PROOF. The result follows directly from the two previous theorems.

7. Maximal $T_{\frac{1}{2}}$ topologies

REMARK. 7.1. Fröhlich [1] defines an ultratopology to be a maximal, non-discrete topology and derives a structure theorem for ultratopologies which is used by Girhinny [2] to prove that each ultratopology is a door space (see Definition 2.4). We can thus prove:

THEOREM 7.2. *$\mathscr{T}$ is a maximal $T_{\frac{1}{2}}$ topology on X iff $\mathscr{T}$ is an ultratopology.*

PROOF. Necessity: If $\mathscr{T}$ is maximal $T_{\frac{1}{2}}$ and $\mathscr{T} \subsetneq \mathscr{U}$, then $\mathscr{U}$ is $T_{\frac{1}{2}}$ by Theorem 5.1, and thus $\mathscr{U}$ is discrete.

Sufficiency: If $\mathscr{T}$ is an ultratopology, then $(X, \mathscr{T})$ is a door space and is $T_{\frac{1}{2}}$ by Corollary 2.7. Hence $\mathscr{T}$ is a maximal $T_{\frac{1}{2}}$ topology on X.

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