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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

By J. Achari.

1. Introduction

Let (M,d) be a metric space. $CB(M)$ stands for non-empty, closed bounded subsets of M, $C(M)$ stands for class of nonempty, compact subsets of M and $CL(M)$ for nonempty closed subsets of M. Let H denote the Hausdorff metric induced by metric d, that is the metric defined by

$$H(A, B)=\inf \{\varepsilon>0;\ A\subset N(B, \varepsilon) \text{ and } B\subset N(A, \varepsilon)\}$$

where

$$N(A,\varepsilon)=\{x\in M;\ d(x,a)<\varepsilon \text{ for some } a\in A\},\ \varepsilon>0,$$ A is a subset of M, and

$$d(x, A)=\inf \{d(x, a);\ a\in A\}$$

In a recent paper Ćirić [1] has proved some fixed point theorems when a mapping T on M satisfies the following inequality.

(1) $\min \{d(Tx, Ty), d(x,Tx), d(y, Ty)\}-\min \{d(x, Ty), d(y,Tx)\}\leq \alpha d(x,y)$

for some $0<\alpha<1$ and all $x,y\in M$. He has also shown that if T is not orbitally continuous then T may fail to have a fixed point.

In this paper we extend the idea of Ćirić to multivalued mappings $F_i (i=1, 2, \cdots m)$ when F_i satisfies the condition

(2) $$\min \{H(F_ix, F_jy), d(x, F_ix), d(y, F_jy)\}-\min \{d(x, F_jy), d(y, F_ix)\} \leq \alpha d(x,y) \quad i, j\in \{1, 2, \cdots m\}$$

for some $0<\alpha<1$ and all $x,y\in M$.

Before going in the theorems we state the following definitions and the result used by Nadler Jr [2].

DEFINITION 1. A multivalued function F_i: $M\rightarrow M$ is a point to set correspondence. An orbit of F_i at the point $x\in M$ is a sequence $\{x_n:\ x_n\in F_n(x_{n-1})\}$, where $x_0=x$. A multivalued function F_i is orbitally upper semicontinuous if $x_n\rightarrow u\in M$ implies $u\in F_iu$ whenever $\{x_n\}$ is an orbit of F_i at some $x\in M$.

DEFINITION 2. A space M is F_i-orbitally complete if every orbit of F_i at some $x \in M$ which is Cauchy sequence, converges in M.

LEMMA 1. *Let $A, B \in CB(M)$. Then for all $\varepsilon > 0$ and $a \in A$, there exists $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$. Furthermore, if $A, B \in C(M)$ then one can select $b \in B$ such that $d(a, b) \leq H(A, B)$.*

2. Fixed Point Theorems

THEOREM 1. *Let M be a F_i-orbitally complete metric space and F_i: $M \to C(M)$, $i=1, 2, \cdots, m$ be orbitally upper semicontinuous mappings satisfing the condition*

(3) $\min \{H(F_i x, F_j y), d(x, F_i x), d(y, F_j y)\} - \min\{d(x, F_j y), d(y, F_i x)\} \leq \alpha d(x, y)$

for all $x, y \in M$, $i, j \in \{1, 2, \cdots, m\}$ and some $0 < \alpha < 1$.

Then $\{F_i\}_{i=1}^{m}$ have a common fixed point. That is there exists $u \in M$ with $u \in F_i u$, $i=1, 2, \cdots, m$.

PROOF. Let now x be arbitrary point in M and let us consider the following orbit of F_i at x

$$x_0 = x,\ x_1 \in F_1(x_0),\ x_2 \in F_2(x_1),\ \cdots,\ x_{m-1} \in F_{m-1}(x_{m-2}),$$
$$x_m \in F_m(x_{m-1}),\ x_{m+1} \in F_1(x_m),\ x_{m+2} \in F_2(x_{m+1}).$$

That is $x_n \in F_r(x_{n-1})$ where $n = qm + r$ with $0 \leq r < m$
and let

$$d(x_n, x_{n+1}) \leq H(F_r(x_{n-1}), F_{r+1}(x_n))$$

We claim that $\{x_n\}$ is Cauchy sequence. It is easy to see that

$$d(x_m, x_{m+1}) \leq H(F_m(x_{m-1}), F_1(x_m))$$

From (3) we get

$$\min \{H(F_m(x_{m-1}), F_1(x_m)), d(F_m(x_{m-1}), x_{m-1}), d(x_m, F_1(x_m))\}$$
$$-\min \{d(x_m, F_m(x_{m-1})), d(x_{m-1}, F_1(x_m)\} \leq \alpha d(x_m, x_{m-1})$$
$$d(x_m, x_{m+1}) \leq \alpha d(x_{m-1}, x_m) \leq \alpha^m d(x_0, F_1 x_0)$$

Next assume by way of induction, that for some integer $p > m$

$$d(x_j, x_{j+1}) \leq \alpha^j d(x_0, F_1 x_0) \quad \text{for } j = 1, 2, \cdots, p-1.$$

Let $p = qm + r$, then $x_{p+1} \in F_{r+1}(x_p)$ and

$$d(x_p, x_{p+1}) \leq H(F_r(x_{p-1}), F_{r+1}(x_p))$$
$$\leq \alpha d(x_p, x_{p-1}) \leq \alpha^p d(x_0, F_1 x_0).$$

This completes induction and thus we have

$$d(x_n, x_{n+1}) \leq \alpha^n d(x_0, F_1 x_0) \quad \text{for } n=1, 2, \cdots\cdots$$

Now

$$d(x_n, x_{n+p}) \leq \sum_{i=n}^{n+p-1} d(x_i, x_{i+1}) \leq \left(\sum_{i=n}^{n+p-1} \alpha_i\right) d(x_0, F_1 x_0) \to 0$$

as $n \to \infty$ for $p=1, 2, \cdots\cdots$

Hence it follows that the orbit of F_i at x is Cauchy sequence. Being M, F_i-orbitally complete, there is some $u \in M$ such that

$$\lim_n x_n = u.$$

Then orbital upper semicontinuity of F_i implies $u \in F_i u$ and this completes the proof of the Theorem.

THEOREM 2. *Let M be a compact metric space and for each $\lambda \in \Lambda$, Λ being an arbitrary indexing set, let F_λ: $M \to CL(M)$ be orbitally upper semicontinous mapping and let*

(4) $\min\{H(F_\lambda x, F_\mu y), d(x, F_\lambda x), d(y, F_\mu y)\} - \min\{d(x, F_\mu y), d(y, F_\lambda x)\} \leq \alpha d(x, y)$ *for all $x, y \in M$, $\lambda, \mu \in \Lambda$ and some $0 < \alpha < 1$. Then the family $\{F_\lambda\}_{\lambda \in \Lambda}$ has a simultaneous fixed points.*

PROOF. Let $B_\lambda = \{x \in M: x \in F_\lambda x\}$ for each $\lambda \in \Lambda$. Then $B_\lambda \neq \phi$. Since F_λ is orbitally upper semicontinous, each B_λ is closed. Next, if B_{λ_i}, $i=1, 2, \cdots m$ is a finite collection, then by Theorem 1 $\bigcap_{i=1}^{m} B_{\lambda_i} \neq \phi$. Thus $\{B_\lambda\}_{\lambda \in \Lambda}$ is a collection of nonempty closed subsets of M having finite intersection property. Using compactness of M, $\bigcap_{\lambda \in \Lambda} B_\lambda \neq \phi$. It is easy to see that for any $u \in \bigcup_{\lambda \in \Lambda} B_\lambda$, $u \in F_\lambda u$ for all $\lambda \in \Lambda$. This completes the proof of the theorem.

Indian Institute of Technology
Kharagpur, India

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