Kyungpook Math. J.
Volume 17, Number 2
December, 1977

ON pc-RINGS

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In this paper rings over which all the cyclic modules are pseudo-injective, called here as pc-rings, are studied and it is shown that

(i) A ring R is left pc iff R/A is left pc for each ideal A of R.

(ii) A left pc-ring is self-pseudo-injective. Moreover, if R is noetherian then R/J is semi-simple artinian where J is the Jacobson radical of R.

(iii) Factor of a pc-ring is self-pseudo-injective. Conversely, if each factor of a duo ring is self pseudo-injective then R is a pc-ring.

(iv) If a prime left Goldie ring R is pc then each quotient of R by a closed ideal is injective.

Throughout this paper R will denote a ring with unit and modules are unitary. $J(R)$ will stand for the Jacobson radical and $Z(R)$ for singular ideal of R. $M\triangle N$ will mean that M is an essential extension of N. An element m of a module M is said to be singular if $R\triangle(0:m)$. The module M is nonsingular if none of its non-zero elements is singular. A ring is said to be left Goldie if it satisfies ascending Chain Condition on annihilator left ideals and does not contain any infinite direct sum of left ideals. An R-module M is said to be pseudo injective if every R-monomorphism of each R-submodule of M into M can be extended to an R-endomorphism of M. A ring R is self-pseudo injective if it is pseudo injective as an R-module.

LEMMA 1. *Let M be an R-module and let A be an ideal of R which annihilates M. Then M is a pseudo-injective R-module iff it is pseudo-injective as an R/A-module.*

PROOF. Trivial, since under the above condition, we have

$$\mathrm{Hom}_R(M, M)=\mathrm{Hom}_{R/A}(M, M).$$

PROPOSITION 2. *If R is a self pseudo-injective ring (with 1) then $J(R)=Z(R)$ and $R/J(R)$ is von Neumann regular.*

PROOF. Suppose $E=\mathrm{Hom}_R(R, R)$. Then since R has 1, the mapping

$$\theta : f \longrightarrow f(1)$$

of E onto R is a ring isomorphism. Under this map $\mathscr{L} \in R$ corresponds to

$$f : x \longrightarrow x\mathscr{L}$$

So, $x \in \ker f \Longleftrightarrow x\mathscr{L} = 0 \Longleftrightarrow x \in (0 : \mathscr{L})$

Hence $\ker f = (0 : \mathscr{L})$.

Now, since R is a pseudo-injective R-module, we have, by [2, Theorem 4.2]:

$$J(E) = \{f \in E / R \triangle \ker f\}.$$

Due to the isomorphism we have

$$J(R) = \{\mathscr{L} \in R / R \triangle (0 : \mathscr{L})\} = Z(R).$$

Again, by the second part of the above cited theorem of [2] we know that $E - J(E)$ is von Neumann regular. It follows that $R/J(R)$ is von Neumann regular in view of the fact that θ maps

$$\{f \in E/R \triangle \ker f\} \text{ into } Z(R)$$

which is shown to be $J(R)$ because of the self-pseudo injectivity of R.

DEFINITION 1. A ring R will be called left(right) *pc*-ring if every left(right) cyclic R-module is pseudo-injective. R is said to be *pc* if it is right and left *pc*.

PROPOSITION 3. *A ring R is left pc iff R/A is left pc for each two sided ideal A of R.*

PROOF. Let R be a left *pc ring* and A an ideal of R. Let I/A be any left ideal of R/A. Consider the R/A-module $(R/A)/(I/A)$. In view of the R-isomorphism

$$(R/A)/(I/A) \cong R/I$$

and the fact that I annihilates the module R/I, A also annihilates the R-module R/I. Therefore R/I may be considered as an R/A-module.

Now, R is left *pc* $\Rightarrow (R/I)$ is R-pseudo-injective. But the ideal A annihilates the R-module (R/I). So, by Lemma 1, R/I considered as an R/A-module is R/A-pseudo-injective. Hence any cyclic R/A-module is R/A-pseudo-injective. R/A is thus a *pc*-ring.

The converse is obvious.

PROPOSITION 4. *Any left pc ring R is self-pseudo-injective. Moreover, if R is noetherian then $R/J(R)$ is semi-simple artinian.*

PROOF. Since R^R is generated by the identity, it is a cyclic left R-module. R^R is therefore, self-pseudo-injective.

Next. self-pseudo-injectivity of R implies von Neumann regularity of $R/J(R)$ (Proposition 2). Moreover, R is noetherian$\Longrightarrow R/J(R)$ is noetherian. Thus, since $R/J(R)$ is noetherian and regular, it is semi-simple artinian.

THEOREM 5. *Factor of a pc-ring R is self-pseudo-injective. Conversely, if each factor of a duo ring R is self-pseudo injetive then R is a pc-ring.*

PROOF. Let A be a left ideal of a *pc* ring R. Then R/A is *pc* by Proposition 3 and hence self pseudo injective by Proposition 4.

Conversely, suppose that each factor ring of R is self-pseudo-injective. Let M be a cyclic R-module. Then $M\cong R/A$ for some left ideal A of R. By assumption, R/A is R/A-pseudo-injective. Hence, by Lemma 1, R/A is R-pseudo-injective. Thus R is *pc*.

PROPOSITION 6. *Let R be a pc ring which is prime left Goldie. Then any quotient of R by a closed ideal is injective.*

PROOF. R is $pc \Rightarrow R/I$ is pseudo-injecive.
Furthermore, R is prime left Goldie implies R is nonsingular.

Now, $Z(R)=0$ and I is closed ideal of R

$\Rightarrow Z(R/I)=0$ [1, Lemma 2.3]

$\Rightarrow R/I$ is torsionfree in Levy's sense [3, Lemma 4.1]

Thus, R/I, being a Levy-torsionfree pseudo-injective module, is injective by [3, Theorem 4.7].

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