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ON MAXIMAL SUBGROUPS OF THE SEMIGROUP OF BINARY RELATIONS

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1. Introduction

A binary relation on a finite set $X=\{x_1, x_2, \cdots, x_n\}$ of n elements is a subset σ of $X\times X=\{(x_i, x_j) ; x_i, x_j\in X\}$. Let B_X be the set of all binary relations on X. (When there is no confusion, an element in B_X is also called a relation on X). Then B_X is a semigroup with the multiplication defined as follows: for $\sigma, \tau\in B_X$, $(x_i, x_j)\in\sigma\tau$ if there is a $x_k\in X$ such that $(x_i, x_k)\in\sigma$ and $(x_k, x_j)\in\tau$. Let θ be the empty relation on X. then θ is the zero element of B_X. Let ω be the universal relation in B_X, *i.e.*, $\omega=X\times X$. In B_X, $\sigma\subset\tau$ means that σ is a subset of τ. Let M_n denote the set of all $n\times n$ matrices over the Boolean algebra of $\{0, 1\}$, then M_n is a semigroup under the ordinary matrix multiplication, and the map

$$\sigma\rightarrow A=(a_{ij}),$$

where

$$a_{ij}=\begin{cases}1 & \text{if } (x_i, x_j)\in\sigma,\\ 0 & \text{otherwise,}\end{cases}$$

is an isomorphism of B_X onto M_n. Let S_X^* (or S_n) be the symmetric group on X, and S_X (or S_n) be the corresponding symmetric group of permutation relations on X, then the map

$$\rho^*\rightarrow\rho$$

is an isomorphism from S_X^* onto S_X where $(x_i, x_j)\in\rho$ if and only if $x_i\rho^*=x_j$. An automorphism of a partially ordered relation $\pi\in B_X$ is a permutation ρ^* on X such that $(x, y)\in\pi$ if and only if $(x\rho^*, y\rho^*)\in\pi$ (if $\leq$ is written for the relation, then this would read as $x\leq y$ if and only if $x\rho^*\leq y\rho^*$). The Montague-Plemmons-Schein theorem in [4] and [5] states that the group of automorphisms of a partially ordered relation $\pi\in B_Y$ where Y is an arbitrary set is isomorphic to the maximal subgroup H_π in B_Y containing π. The results in [2] can be stated as follows:

THEOREM MPS. 1. *The group of automorphisms of π denoted by G_π is* $\{\rho\in S_Y; \pi\rho=\rho\pi\}$.

2. *The maximal subgroup H_π in B_Y containing π is* $\{\rho\pi ; \rho\in G_\pi\}$.

3. $G_\pi\simeq H_\pi$ *under the map* $\rho\rightarrow\rho\pi(=\pi\rho)$.

Here, we consider the following problem: Given a group G^* of permutations on a finite set X regarded as a group of permutation binary relations G in B_X, can we find a partially ordered relation $\pi\in B_X$ such that $G\pi=\{\rho\pi ; \rho\in G\}$ is the maximal subgroup H_π in B_X containing π? In 2, we shall present a way to partition the universal relation ω in B_X, and to partition B_X. The former leads to an algorithm, in 3, for constructing all partially ordered relations π each whose maximal subgroup $H_\pi\supset G\pi$. The later determines the number of isomorphic relations in B_X for any given relation in B_X. In 4, an example is given to demonstrate the algorithm. If G is any given abstract group, then, by Birkhoff's theorem in [1], there exists a partially ordered set whose group of automorphisms is isomorphic to G. However, if G^* is a given group of permutations of X, there may not exist a partially ordered relation on X whose group of automorphisms is G. In general, it seems to be very difficult to determine which permutation group G^* on X can have a partially ordered relation π in B_X whose group of automorphisms is G^*, (or whose $H_\pi=G\pi$). In 5, we present a result concerning this problem.

2. Partitions

Let G^* be a permutation group on X, and G be the group of permutation relations in B_X corresponding to G^*. We shall consider two partitions:

(1) Partition ω with respect to G^*, and

(2) Partition B_X with respect to S_X^*.

Let $\sigma_{ij}=\{(x_i, x_j)\}\in B_X$ for $i, j=1, 2, \cdots, n$.

Then
$$\omega=\bigcup_{i,j}\sigma_{ij}.$$

Clearly, every member in B_X is a union of some σ_{ij}'s. Two relations σ_{ij} and σ_{kl} are said to be similar with respect to G, denoted by $\sigma_{ij}R\sigma_{kl}$, if and only if there exists a $\rho\in G$ such that
$$\rho\sigma_{ij}\rho^{-1}=\sigma_{kl}.$$

Since G_X is a group, this similarity is an equivalence relation. Consequently, with respect to this similarity, ω is partitioned into disjoint subsets $\gamma^{(1)}$, $\gamma^{(2)}$, $\cdots$, $\gamma^{(m)}$. Also, $\sigma_{ij}R\sigma_{kl}$ if and only if there exists a $\rho^*\in G^*$ such that $x_i\rho^*=x_k$ and $x_j\rho^*=x_l$. Let $Z(G)=\{\mu\in B_X : \mu\rho=\rho\mu$ for all $\rho\in G\}$. Clearly, $Z(G)$ is a subsemigroup of B_X.

THEOREM 1. (a) $\gamma^{(s)}\in Z(G)$ *for* $s=1, 2, \cdots, m$.

(b) *If* $\mu\in Z(G)$ *then* μ *is a union of some of* $\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(m)}$.

PROOF. Since B_X and M_n are isomorphic, we write a relation $\tau\in B_X$ as a matrix $\tau=(\tau_{ij})$ or $\tau=(\tau)_{ij}$, $i, j=1, 2, \cdots, n$, *i.e.*, $\tau_{ij}=1$ if $(x_i, x_j)\in\tau$, and $\tau_{ij}=0$ if $(x_i, x_j)\notin\tau$. Since each $\rho=(\rho_{ij})\in G$ is a permutation relation in B_X, its group inverse is also its converse relation. Hence, for every $\rho\in G$, we have

$$(\rho\gamma^{(s)}\rho^{-1})_{ij}=\sum_{k=1}^{n}\sum_{t=1}^{n}\rho_{it}\gamma^{(s)}_{tk}\rho_{jk}=\rho_{ir}\gamma^{(s)}_{ru}\rho_{ju}.$$

If $\gamma^{(s)}_{ru}=1$, then $\sigma_{ru}\subset\gamma^{(s)}$. Since $\rho_{ir}=\rho_{ju}=1$, there exists $\rho^*\in G^*$ such that $x_i\rho^*=x_r$ and $x_j\rho^*=x_u$. This means that $\sigma_{ij}R\sigma_{ru}$, $\sigma_{ij}\subset\gamma^{(s)}$ and $\gamma^{(s)}_{ij}=1$. If $\gamma^{(s)}_{ru}=0$, then $\sigma_{ru}\not\subset\gamma^{(s)}$, $\sigma_{ij}\not\subset\gamma^{(s)}$ and $\gamma^{(s)}_{ij}=0$. Hence,

$$(\rho\gamma^{(s)}\rho^{-1})_{ij}=\gamma^{(s)}_{ij}$$

for all $i, j=1, 2, \cdots, n$, *i.e.*, for every $\rho\in G$, $\rho\gamma^{(s)}=\gamma^{(s)}\rho$ and $\gamma^{(s)}\in Z(G)$ for $s=1, 2, \cdots, m$.

(b) Let $\mu=(\mu_{ij})$ be an arbitrary relation in $Z(G)$. Since $\rho\mu\rho^{-1}=\mu$ for every $\rho\in G$, $\mu_{ij}=\mu_{(i\rho^*)(j\rho^*)}$ where ρ^* runs through G^*. For $\mu_{ij}=1$, there is a $\gamma^{(s)}$ such that $\gamma^{(s)}_{ij}=1$, $1\leq s\leq m$, since $\omega=\bigcup_{v=1}^{m}\gamma^{(v)}$. By (a), we know each $\gamma^{(s)}\in Z(G)$ which implies $\gamma^{(s)}_{ij}=\gamma^{(s)}_{(i\rho^*)(j\rho^*)}$ for every $\rho^*\in G^*$, and $\gamma^{(s)}\subset\mu$. If $\gamma^{(s)}=\mu$, then our proof is completed. If $\gamma^{(s)}\subset\mu$, then for some k and q we have $\mu_{kl}=1$ and $\gamma^{(s)}_{kq}\neq 1$. Again, by $\omega=\bigcup_{V=1}^{m}\gamma^{(V)}$, there is a $\gamma^{(u)}$ such that $\gamma^{(u)}_{kq}=1$, $1\leq u\leq m$, and we repeat the similar argument to obtain $\gamma^{(s)}\cup\gamma^{(u)}\subset\mu$. Repeating the similar procedure for at most m times, we obtain μ as a union of some of $\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(m)}$.

For the second partition, we have the following: Let $\sigma, \tau\in B_X$. σ and τ are said to be isomorphic, denoted by $\sigma R'\tau$, if and only if there exists a permutation relation $\rho\in S_X\subset B_X$ such that

$$\rho\sigma\rho^{-1}=\tau$$

where S_X is the symmetric group of all permutation relatons in B_X. (Since it was shown in [3] that every automorphism of B is inner and the group of automorphisms of B_X is isomorphic to S_X, it is justified to say that σ and τ are isomorphic). Again, since S_X is a group, the isomorphic relation on B_X is an equivalence relation, and B_X is partitioned into disjoint subsets. Let $\sigma\in B_X$, and $C_\sigma=\{\rho\in S_X;\ \rho\sigma=\sigma\rho\}$. Then C_σ is a group, Also, let $|C_\sigma|$ denote the cardinality of C_σ. (Our C_σ is G_σ in [2]).

THEOREM 2. *Let $\alpha, \beta\in B_X$, $\alpha\subset\beta$ and $C_\alpha\subset C_\beta$. Then the number of γ in B_X, such that $\gamma R'\alpha$ and $\gamma\subset\beta$, is $\geq$ the index of C_α in C_β.*

PROOF. We claim that the elements in the left coset μC_α transform α alike in β, and the elements in the different left cosets transform α differently in β. Let $\mu\rho, \mu\tau\in\mu C_\alpha$, then

$$(\mu\rho)\alpha(\mu\rho)^{-1}=\mu(\rho\alpha\rho^{-1})\mu^{-1}=\mu\alpha\mu^{-1},$$

and

$$(\mu\tau)\alpha(\mu\tau)^{-1}=\mu(\tau\alpha\tau^{-1})\mu^{-1}=\mu\alpha\mu^{-1},$$

i.e., they transform α alike. In fact, they transform α alike in β, since $\alpha\subset\beta$ and $\mu\in C_\beta$. Also, let μC_α and ηC_α be different cosets, and $\mu\rho\in\mu C_\alpha$ and $\eta\tau\in\eta C_\alpha$. Suppose $\mu\rho$ and $\eta\tau$ transfoming α alike, then we would have

$$\mu\alpha\mu^{-1}=\eta\alpha\eta^{-1},$$

i.e., $\eta^{-1}\mu\in C_\alpha$, $\mu\in\eta C_\alpha$ and $\mu C_\alpha\subset\eta C_\alpha$. That is a contradiction. Hence, the number of γ, such that $\gamma R'\alpha$ and $\gamma\leq\beta$, is $\geq$ the index of C_α in C_β.

COROLLARY 2.1. *Let $\alpha\in B_X$. Then the number of γ in B_X, such that $\gamma R'\alpha$, is equal to $n!/|C_\alpha|$.*

PROOF. Let $\alpha\subset\omega$. Since $C_\omega=S_X$ and $|S_X|=n!$, apply Theorem 2 to complete the proof.

3. An algorithm

Given a permutation group G^* on X, we shall find all partially ordered relations π on X such that the maximal subgroup H_π in B_X containing π must $\supset G_\pi$. Let Γ be this collection, *i.e.*, $\Gamma=\{\pi\in B_X;\ \pi$ is a partially ordered relation and $H_\pi\supset G_\pi\}$. The algorithm for obtaining Γ goes as follows:

(1) From G^* we can obtain the collection $\{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(m)}\}$ as in Theorem 1, and we denote this collection by Γ'.

(2) Let Γ'' be the subcollection of Γ' such that each member of Γ'' does not contain any cycle of length greater than 1. (*e.g.*, σ_{ii} is a cycle of length 1, and $\sigma_{ij}\cup\sigma_{jk}\cup\sigma_{ki}$, where i, j and k are pairwise different, is a cycle of length3). Let $\delta=\bigcup_{i=1}^{n}\sigma_{ii}$. Say, $\Gamma''=\{\delta, \gamma^{(i_1)}, \gamma^{(i_2)}, \cdots, \gamma^{(i_t)}\}$.

(3) Let $\Gamma'''=\{\delta, \delta\cup\gamma^{(i_1)}, \delta\cup\gamma^{(i_2)}, \cdots, \delta\cup\gamma^{(i_t)}\}$, and Γ be the set of all possible unions of members of Γ''' with the following conditions: (a) None of members of Γ contains a cycle of length greater than 1, and (b) If a member, π, of Γ contains σ_{ij} and σ_{jk} where i, j and k are pairwise different, then π must also contain σ_{ik}. Since the cardinality of Γ' is finite, it takes only a finite of steps to obtain Γ.

We claim that the algorithm gives us all partially ordered relations π on X each whose H_π in $B_X\supset G\pi$: Let π be a relation on X which is obtained by the algorithm, then, by (3), π is a union of members of Γ''' and π is a partially ordered relation on X. By Theorem 1, each $\gamma^{(i)}\varepsilon Z(G)$. Since π is a union of some of $\gamma^{(i)}$'s, $\pi\in Z(G)$. By Theorem MPS, we know $G\pi\subset H_\pi$. Conversely, let π be a partially ordered relation on X whose H_π in $B_X\supset G\pi$. Then, by Theorem MPS, $\pi\in Z(G)$, and, by Theorem 1, π is a union of some $\gamma^{(i)}$'s in Γ'. Since π is a partially ordered relation on X, π contains δ and π satisfies the conditions (a) and (b) in (3). Hence, $\pi\in\Gamma$, *i.e.*, π must be one of the relations obtained by using the algorithm.

4. An example

For convenience, we let $X=\{1, 2, \cdots, 6\}$ instead of $X=\{x_1, x_2, \cdots, x_6\}$. Also, we let G^* be the permutation group on X generated by

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6\\ 1 & 3 & 2 & 5 & 6 & 4\end{pmatrix}.$$

Hence $|G^*|=6$. We shall construct all of the partially ordered relations π on X each whose maximal subgroup H_π in B_X containing $\pi\supset G\pi$:

Since $\sigma_{ij}R\sigma_{(i\rho^*)(j\rho^*)}$ for all $\rho^*\in G^*$, we have $\Gamma'=\{\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(12)}\}$ where

$\gamma^{(1)}=\sigma_{11}$, $\gamma^{(2)}=\sigma_{22}\cup\sigma_{33}$, $\gamma^{(3)}=\sigma_{44}\cup\sigma_{55}\cup\sigma_{66}$,

$\gamma^{(4)}=\sigma_{12}\cup\sigma_{13}$, $\gamma^{(5)}=\sigma_{14}\cup\sigma_{15}\cup\sigma_{16}$,

$\gamma^{(6)}=\sigma_{21}\cup\sigma_{31}$, $\gamma^{(7)}=\sigma_{41}\cup\sigma_{51}\cup\sigma_{61}$,

$\gamma^{(8)}=\sigma_{24}\cup\sigma_{25}\cup\sigma_{26}\cup\sigma_{34}\cup\sigma_{35}\cup\sigma_{36}$,

$\gamma^{(9)}=\sigma_{42}\cup\sigma_{43}\cup\sigma_{52}\cup\sigma_{53}\cup\sigma_{62}\cup\sigma_{63}$,

$\gamma^{(10)}=\sigma_{23}\cup\sigma_{32}$, $\gamma^{(11)}=\sigma_{45}\cup\sigma_{56}\cup\sigma_{64}$ and

$\gamma^{(12)}=\sigma_{46}\cup\sigma_{54}\cup\sigma_{65}$.

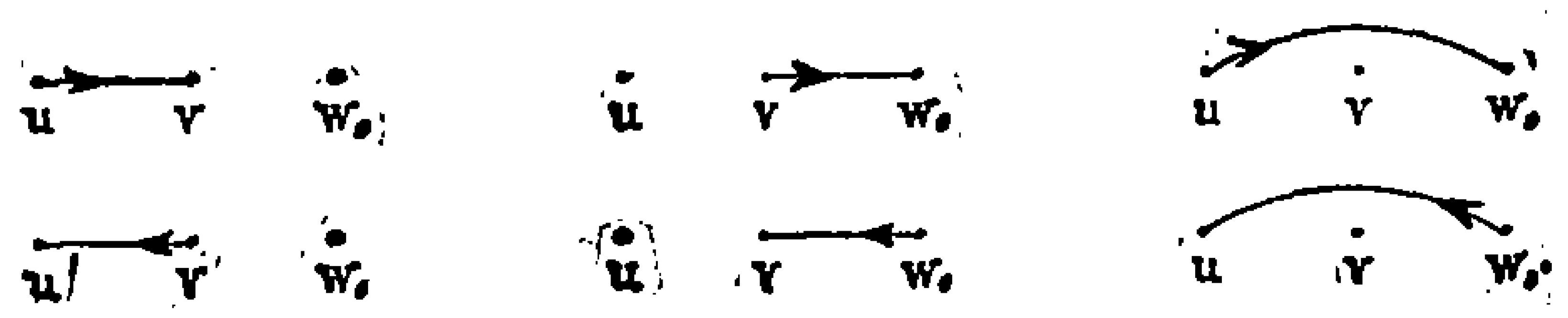
In matrix notation, we have

$$Z(G)=\left\lceil \begin{pmatrix} \frac{a}{f} & \Big| \frac{d}{b} & \frac{d}{j} & \Big| \frac{e}{h} & \frac{e}{h} & \frac{e}{h} \\ \frac{f}{g} & \Big| \frac{j}{i} & \frac{b}{i} & \Big| \frac{h}{c} & \frac{h}{k} & \frac{h}{l} \\ g\,|\,i & & i\,|\,l & & c & k \\ g\,|\,i & & i\,|\,k & & l & c \end{pmatrix} ;\ a,\ b,\ \cdots,\ l\in\{0,\ 1\} \right\rceil.$$

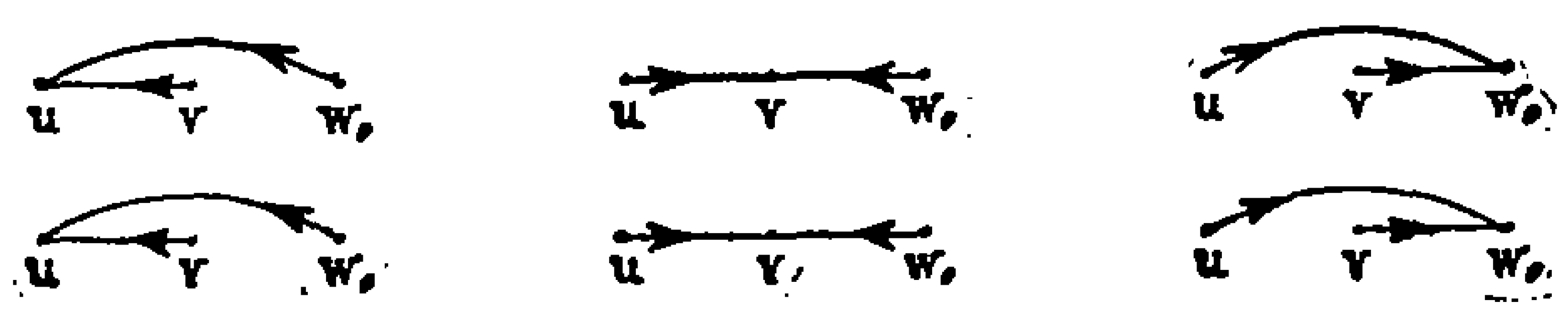
Since each of $\gamma^{(10)}$, $\gamma^{(11)}$ and $\gamma^{(12)}$ contains a cycle of length greater than 1, we delete them. Hence, $\Gamma''=\{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}, \gamma^{(5)}, \gamma^{(6)}, \gamma^{(7)}, \gamma^{(8)}, \gamma^{(9)}\}$

Let $\delta=\gamma^{(1)}\cup\gamma^{(2)}\cup\gamma^{(3)}$. Then $\Gamma'''=\{\delta, \delta\cup\gamma^{(4)}, \delta\cup\gamma^{(5)}, \delta\cup\gamma^{(6)}, \delta\cup\gamma^{(7)}, \delta\cup\gamma^{(8)}, \delta\cup\gamma^{(9)}\}$. By using the conditions (a) and (b) in (3) of our algorithm, we can list the members of Γ with the help of the graphs: Let u, v and w be the vertices of a graph where $u=\{1\}$, $v=\{2,3\}$ and $w=\{4,5,6\}$. If there is a directed edge from v to w, it means that every element in v is related to every element in w, and if there is no edge between v and w, it means that none of the elements in v is related to any element in w, We count graphs without drawing the loops, but with no edge:

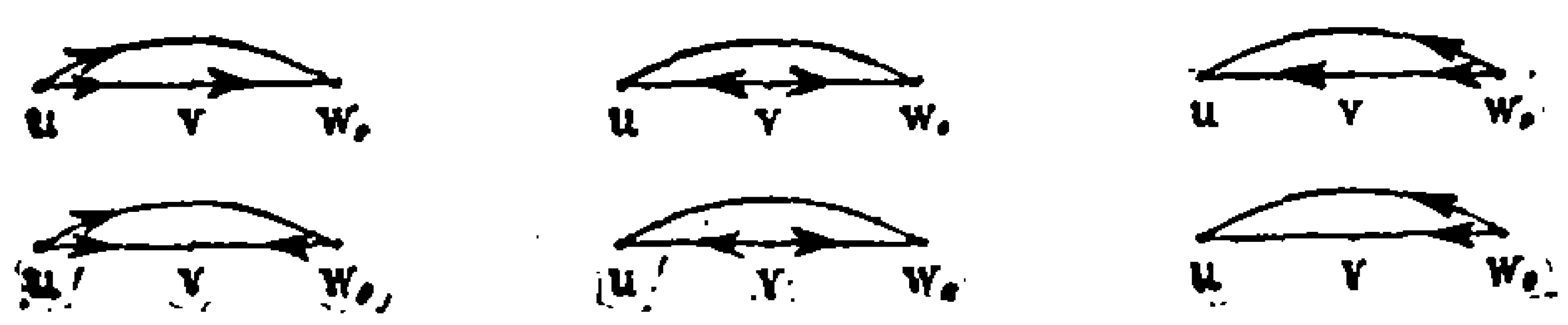
one edge:



two edges:



three edges:



Since more than three edges would create a cycle of length greater than 1, these are all possible cases. The corresponding partially ordered relations constitute Γ (π_{13} corresponds the third graph with one edge):

$$\pi_{01}=\delta$$

$$\pi_{11}=\delta\cup(\sigma_{12}\cup\sigma_{13})=\delta\cup\gamma^{(4)},$$

$$\pi_{12}=\delta\cup(\sigma_{24}\cup\sigma_{25}\cup\sigma_{26}\cup\sigma_{34}\cup\sigma_{35}\cup\sigma_{36})=\delta\cup\gamma^{(8)},$$

$$\pi_{13}=\delta\cup(\sigma_{14}\cup\sigma_{15}\cup\sigma_{16})=\delta\cup\gamma^{(5)},$$

$$\pi_{14}=\delta\cup(\sigma_{21}\cup\sigma_{31})=\delta\cup\gamma^{(6)},$$

$$\pi_{15}=\delta\cup(\sigma_{42}\cup\sigma_{43}\cup\sigma_{52}\cup\sigma_{53}\cup\sigma_{62}\cup\sigma_{63})=\delta\cup\gamma^{(9)},$$

$$\pi_{16}=\delta\cup(\sigma_{41}\cup\sigma_{51}\cup\sigma_{61})=\delta\cup\gamma^{(7)},$$

$$\pi_{21}=\pi_{11}\cup\pi_{13}=\delta\cup\gamma^{(4)}\cup\gamma^{(5)},$$

$$\pi_{22}=\pi_{12}\cup\pi_{14}=\delta\cup\gamma^{(8)}\cup\gamma^{(6)},$$

$\pi_{23}=\pi_{15}\cup\pi_{16}=\delta\cup\gamma^{(9)}\cup\gamma^{(7)}$,

$\pi_{24}=\pi_{14}\cup\pi_{16}=\delta\cup\gamma^{(6)}\cup\gamma^{(7)}$,

$\pi_{25}=\pi_{15}\cup\pi_{11}=\delta\cup\gamma^{(9)}\cup\gamma^{(4)}$,

$\pi_{26}=\pi_{13}\cup\pi_{12}=\delta\cup\gamma^{(5)}\cup\gamma^{(8)}$,

$\pi_{31}=\pi_{11}\cup\pi_{12}\cup\pi_{13}=\delta\cup\gamma^{(4)}\cup\gamma^{(8)}\cup\gamma^{(5)}$,

$\pi_{32}=\pi_{14}\cup\pi_{12}\cup\pi_{16}=\delta\cup\gamma^{(6)}\cup\gamma^{(8)}\cup\gamma^{(7)}$,

$\pi_{33}=\pi_{14}\cup\pi_{15}\cup\pi_{16}=\delta\cup\gamma^{(6)}\cup\gamma^{(9)}\cup\gamma^{(7)}$,

$\pi_{34}=\pi_{11}\cup\pi_{15}\cup\pi_{13}=\delta\cup\gamma^{(4)}\cup\gamma^{(9)}\cup\gamma^{(5)}$,

$\pi_{35}=\pi_{14}\cup\pi_{12}\cup\pi_{16}=\delta\cup\gamma^{(6)}\cup\gamma^{(8)}\cup\gamma^{(7)}$, and

$\pi_{36}=\pi_{11}\cup\pi_{15}\cup\pi_{16}=\delta\cup\gamma^{(4)}\cup\gamma^{(9)}\cup\gamma^{(7)}$.

In matrix notation, *e.g.*, $\pi_{11}$and π_{36}, respectively, are:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Also, $H_{\pi_{01}}=S_6\supset G_{\pi_{01}}$, and $H_{\pi_{ij}}=S_1xS_2xS_3\supset G_{\pi_{ij}}$ for $i=1, 2, 3$ and $j=1, 2, \cdots, 6$.

5. Automorphisms

We know that every partially ordered relation π in B_X has a group of automorphisms. Does the converse hold? That is, given a permutation group G^* on X, does there exist a $\pi\in B_X$ whose group of automorphisms is G^* (or $H_\pi=G\pi$) ? From the previous example, we see that there does not exist any $\pi\in B_X$ whose group of automorphisms is the given group. To find a necessary and sufficient condition(s) seems to be very difficult. Here we have:

THEOREM 3. *Let G^* be a permutation group on X, and $X^{(1)}, X^{(2)}, \cdots, X^{(q)}$ be its orbits,* $1\leq q\leq n$. *If G restricted on $X^{(i)}$is $S_{X^{(i)}}$ for $i=1, 2, \cdots, 1$, i.e., q, i.e.,* $G|_{X^{(i)}}=S_{X^{(i)}}$, *and if G^* is isomorphic to the direct product of* $S_{X^{(1)}}, S_{X^{(2)}}, \cdots$,

$S_{X^{(q)}}$, *then there exists a partially ordered relation* π *in* B_X *such that* G^* *is the group of automorphisms* π *and* $H_\pi = G\pi$.

PROOF. We construct a partially ordered relation π in B_X as follows: For $q=1$, we have $G=S_X$. Let $\pi=\{(x,x) : x\in X\}$. Then, clearly, π is a partially ordered relation in B_X, $\rho\pi=\pi\rho$ for every $\rho\in G=S_X$. By Theorem MPS, G^* is the group of automorphisms of π and $H_\pi=G\pi$.

For $q>1$, let $\pi=\{(x, x) ; x\in X\}\cup\{(x_i, x_j) ; x_i\in X^{(i)}, x_j\in X^{(j)}, i<j, i=1, 2, \cdots, q-1$ and $j=2, 3, \cdots, q\}$. Then one can easily verify that π is a partially ordered relations in B_X, and ρ^* preserves the order relation for every $\rho^*\in G^*$, *i.e.*, $\rho\pi=\pi\rho$ for every $\rho\in G$, and G^* is contained in the group of automorphisms of π. Let $\tau\in S_X$ and $\tau\pi=\pi\tau$. We claim that $\tau\in G$. Since the number of relations of each $x_i\in X^{(i)}$ differs with the number of relations of each $x_j\in X^{(j)}$ for $i\neq j$, τ^* cannot map an element in $X^{(i)}$ to an element in $X^{(j)}$ for $i\neq j$, but τ^* can map an element in $X^{(i)}$ to any other element in $X^{(i)}$. Since G^* is isomorphic to the direct product of $S_{X^{(1)}}, S_{X^{(2)}}, \cdots, S_{X^{(q)}}$, $\tau\in G$. By Theorem MPS, G^* is the group of automorphisms of π, and $H_\pi=G\pi$,

THEOREM 4. *Let* π *be a partially ordered relation in* B_X *and* G^* *be its group of automorphisms. If the lengths of orbits of* G^*, $|X^{(i)}|$ *for* $i=1, 2, \cdots, q$ *and* $1\leq q\leq n$, *are distinct primes* (1 *may be considered as a prime here*), *then* $G|_{X^{(i)}}=S_{X^{(i)}}$, $i=1, 2, \cdots, q$, G^* *is isomorphic to the direct product of* $S_{X^{(1)}}, S_{X^{(2)}}, \cdots, S_{X^{(q)}}$, *and* $H_\pi=G\pi$.

PROOF. For $q=1$, a partially ordered relation π in B_X with a transitive group of automorphisms implies $\pi=\{(x, x) ; x\in X\}$, and $G^*=S_X$. By Theorem 3. $H_\pi=G\pi$. For $q>1$, let $X^{(i)}$ and $X^{(j)}$ be any two distinct orbits of G^*. Say, $|X^{(i)}|=p_i$ and $|X^{(j)}|=p_j$. Since a transitive permutation group on a prime number p of points contains an element of order p, $G^*|_{X^{(i)}}$ and $G^*|_{X^{(j)}}$ contain, respectively, an element of order p_i and an element of order p_j. Also, $G^*|_{X^{(i)}\cup X^{(j)}}$ contains an element p^* of order p_ip_j. Let $x_i\in X^{(i)}$ and $x_j\in X^{(j)}$, then either $(x_i, x_j)\notin\pi$, or $(x_i, x_j)\in\pi$. If $(x_i, x_j)\in\pi$, then $(x_i(\rho^*)^k, x_j(\rho^*)^k)\in\pi$ for $k=1, 2, \cdots, p_ip_j$, *i.e.*, $(x_i, x_j)\in\pi$ for every $x_i\in X^{(i)}$ and $x_j\in X^{(j)}$. Consequently, a transposition belongs to $G^*|_{X^{(i)}}$. Since G^* restricted on a prime number p_i points contains an element of order p_i and a transposition, $G^*|_{X^{(i)}}=S_{X^{(i)}}$. It follows that G^* is isomorphic to the direct product of $S_{X^{(1)}}$, $S_{X^{(2)}}$, $\cdots$, $S_{X^{(t)}}$. By Theorem 3, $H_\pi = G\pi$.

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