Kyungpook Math. J.
Volume 17, Number 2
December, 1977

DIRECT SUMS OF SEMIRINGS AND THE KRULL-SCHMIDT THEOREM

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1. Introduction

P. Allen and W. Windham in [1] discussed briefly commutative semirings with the property that any two ideals are related. This property of ideals being related turns out to be a key to generalizing direct sums of semirings. The purpose of this paper is to prove necessary and sufficient conditions for a semiring to be a direct sum of ideals and to prove an analogue of the Krull-Schmidt theorem for semirings.

2. Basic concepts

DEFINITION 2.1. A set R together with two binary operations $(+)$ and $(\cdot)$ is called *a semiring* provided; (1) $(R,+)$ is a commutative semigroup with a zero, (2) $(R,\cdot)$ is a semigroup, and (3) $(\cdot)$ distributes over $(+)$ from both the left and the right. If $(R,\cdot)$ is a commutative semigroup, R is called a commutative semiring. If there is an elemnt $e\in R$ such that $er=re=r$ for all $r\in R$ then R is called a semiring with an identity.

DEFINITION 2.2. A non-empty subset H of a semiring R is called *an ideal* in R if, (1) H is closed under $(+)$, and (2). $HR\subset H$ and $RH\subset H$.

DEFINITION 2.3. Let R_1 and R_2 be semirings. A mapping η: $R_1\to R_2$ is called *a homomorphism* if $(a+b)\eta=a\eta+b\eta$ and $(ab)\eta=(a\eta)(b\eta)$ for all a, $b\in R$. As usual, η is called an isomorphism if η is both one-to-one and onto.

DEFINITION 2.4. A homomorphism η: $R_1\to R_2$ is called *semimaximal* if $r_1\eta=r_2\eta$ implies that $r_1+\ker\eta\cap r_2+\ker\eta\neq\phi$.

DEFINITION 2.5. A homomorphism $\eta:R\to R$ will be called *complete* if η is semimaximal and there is an integer $t\geq 1$ such that $R=R\eta^t+\ker\eta^t$.

DEFINITION 2.6. Let R be a semiring and H_1 and H_2 ideals in R. H_1 is said to be *related to* H_2 if $h_1+h_2=h_1'+h_2'$, h_i, $h_i'\in H_i$, implies that there are a, $b\in H_1\cap H_2$ such that $h_1+a=h_1'+b$.

EXAMPLE 2.7. Let R be the set of non-negative integers$\cup\{\infty\}$. Define $a+b=\max\{a, b\}$ and $ab=\min\{a, b\}$. Clearly R is a commutative semiring with an identity and any two ideals in R are related, since an ideal in R has the form $H=\{a\in R\,|\,a<r\}$, where $r\in R$. Now let H be an ideal in R and define η: $R\rightarrow R$ by $\eta(a)=0$ if $a\in H$ and $\eta(a)=a$ if $a\in R-H$. It is clear that η is a homomorphism and $\ker\eta=H$. Now $R\eta=\{R-H\}\cup\{0\}$. Consequently, $R=R\eta+ker\eta$ and η is complete.

3. Direct sums of semirings

Let $R_1, R_2, \cdots, R_n$ be n commutative semirings with identities $e_1, e_2, \cdots, e_n$ respectively. Consider the set $R=\{(a_1, a_2, \cdots, a_n)\,|\,a_i\in R_i\}$. In R, define $(a_1, a_2, \cdots, a_n)=(b_1, b_2, \cdots, b_n)$ if and only if $a_i=b_i$ for $1\leq i\leq n$. Further, for $a=(a_1, \cdots, a_n)$and $b=(b_1, \cdots, b_n)$, define $a+b=(a_1+b_1, \cdots, a_n+b_n)$and $ab=(a_1b_1, \cdots, a_nb_n)$. It is easily checked that R is a commutative semiring with identity $e=(e_1, \cdots, e_n)$. The semiring R is called a direct sum of the semiring $R_1, \cdots, R_n$ and is denoted by $R=R_1\oplus R_2\oplus\cdots\oplus R_n$.

THEOREM. 3.1. *A semiring R is isomorphic to a direct sum $R_1\oplus R_2\oplus\cdots\oplus R_n$ if and only if R contains ideals S_i, each with an identity, such that*

(1) $R=\sum S_i$, (2) $S_i\cap(\sum_{i\neq j} S_j)=0$ *and* (3) S_i *is related to* $(\sum_{i\neq j} S_j)$.

PROOF. Let $R=R_1\oplus\cdots\oplus R_n$ and $S_i=\{a_i'=(0, \cdots, 0, a_i, 0, \cdots, 0)\,|\,a_i\in R_i\}$. It is easily checked that S_i is a subsemiring of R with identity $e_i'=(0, \cdots, e_i, \cdots, 0)$, and that S_i is an ideal in R. Now from $x\in R$, it follows that $x=(a_1, a_2, \cdots, a_n)=(a_1, 0, \cdots, 0)+(0, a_2, 0, \cdots, 0)+\cdots+(0, \cdots, 0, a_n)=a_1'+a_2'+\cdots+a_n'$. Consequently $R=\sum S_i$. Next sppose that $x\in S_i\cap(\sum_{i\neq j} S_j)$. Then $(0, \cdots, a_i, \cdots 0)=x=(a_1, \cdots, a_{i-1}, 0, a_{i+1}, \cdots a_n)$ and it follows that $a_i=0$ for each i. Consequently $x=0$ and $S_i\cap(\sum_{i\neq j} S_j)=0$. Now suppose that a_i' and $b_i'\in S_i$, x and $y\in(\sum_{i\neq j} S_j)$ and $a_i'+x=b_i'+y$. Hence $(x_1, \cdots, x_{i-1}, a_i, x_{i+1}, \cdots, x_n)=(y_1, \cdots, y_{i-1}, b_i, y_{i+1}, \cdots, y_n)$. Thus $x_j=y_j$ for each j and $a_i=b_i$. But $S_i\cap(\sum_{i\neq j} S_j)=0$. So $a_i'+0=b_i'+0$ and S_i is related to $(\sum_{i\neq j} S_j)$. Conversely, suppose that R is a semiring containing ideals S_i such that (1), (2), and (3) hold. Let $S=S_1\oplus\cdots\oplus S_n$ and define a mapping ϕ: $S\rightarrow R$ *by* $\phi(a_1, \cdots, a_n)=a_1+a_2+\cdots+a_n$. To show that ϕ is an isomorphism. First, $\phi[(a_1, \cdots, a_n)+(b_1, \cdots, b_n)]=\phi[(a_1+b_1, \cdots, a_n+b_n)]=a_1+b_1+\cdots+a_n+b_n=(a_1+a_2+\cdots+a_n)+(b_1+\cdots+b_n)=\phi(a_1, \cdots, a_n)+\phi(b_1, \cdots, b_n)$.

Next observe that $a_ia_j=0$ if $i\neq j$. To see this note that $a_ia_j\in S_j$ and $a_ia_j\in S_i$. But $S_i\cap S_j=0$ if $i\neq j$. Consequently $a_ia_j=0$. Thus $\phi[(a_1, \cdots, a_n)(b_1, \cdots, b_n)]=\phi[a_1b_1, \cdots, a_nb_n]=a_1b_1+\cdots+a_nb_n$. and $\phi(a_1, \cdots, a_n)\phi(b_1, \cdots, b_n)=(a_1+\cdots+a_n)\cdot(b_1+\cdots+b_n)=a_1b_1+a_2b_2+\cdots a_nb_n$. Consequently $\phi[(a_1 \cdots, a_n)(b_1, \cdots, b_n)]=\phi(a_1, \cdots, a_n)\cdot\phi(b_1, \cdots, b_n)$ and ϕ is a semiring homomorphism. Now let $x\in R$. By (1), $x=a_1+\cdots+a_n$ where $a_i\in S_i$. Consequently, $(a_1, \cdots, a_n)\in S$, $\phi(a_1, \cdots, a_n)=x$ and ϕ is onto. Next suppose that $\phi(a_1, \cdots, a_n)=\phi(b_1, \cdots, b_n)$. Then $a_1+a_2+\cdots+a_n=b_1+\cdots+b_n$. Since S_i is related to $(\sum_{i\neq j} S_j)$ and $S_i\cap(\sum_{i\neq j} S_i)=0$, it follows that $a_i=b_i$ for each i. Therefore $(a_1, \cdots, a_n)=(b_1, \cdots, b_n)$ and ϕ is one to one. Consequently, ϕ is an isomorphism and the proof is complete.

THEOREM. 3.2. *If R is a semiring and R contains ideals R_i such that each $x\in R$ has a unique representation $x=x_1+\cdots+x_n$ where $x_i\in R_i$, then $R=R_1\oplus\cdots\oplus R_n$.*

PROOF. From $x=x_1+\cdots+x_n$, it follows that $R=\sum R_i$. Now suppose $x\in R_i\cap(\sum_{i\neq j} R_j)$. Then $x=x_i$ and $x=x_1+\cdots+x_{i-1}+0+x_{i+1}+\cdots+x_n$. But the representation for x is unique. Consequently, $x_i=0$ for each i. Thus $x=0$ and $R_i\cap\sum_{i\neq j}R_j=0$. If a and $b\in R_i$, x and $y\in(\sum_{i\neq j} R_j)$, and $a+x=b+y$, then it follows from uniqueness of representation and $R_i\cap(\sum_{i\neq j} R_j)=0$ that $a=b$. Thus R_i is related to $(\sum_{j\neq i} R_j)$. Therefore $R=R_1\oplus\cdots\oplus R_n$ by theorem 3.1.

If $R_1, \cdots, R_n$ are semirings, then $R=R_1\oplus\cdots\oplus R_n$ is called an external direct sum. If R is semiring $S_1, \cdots, S_n$ are ideals in R such that $R=S_1\oplus\cdots\oplus S_n$, then R is called an internal direct sum.

4. Projections

Let $R=R_1\oplus\cdots\oplus R_n$ be an internal direct sum of semirings and define ε_i: $R\to R_i$ by $x\varepsilon_i=x_i$ for $x=x_1+\cdots+x_n$. It is clear that ε_i is a semiring homomorphism. If $x_i\in R_i$, then $x_i\varepsilon_i=x_i$ If $x_j\in R_j$ and $i\neq j$, then $x_j\varepsilon_i=0$. Also $x\varepsilon_i^2=x_i\,\varepsilon_i=x_i$ and $\varepsilon_i=\varepsilon_i^2$. Consequently, ε_i is idempotent. Denote the endomorphism $x\to 0$ for all $x\in R$ by 0_R and the identity endomorphism by 1_R. Then it is clear that $\varepsilon_i\varepsilon_j=0_R$ if $i\neq j$. Now suppose $x=x_1+\cdots+x_n$, $y=y_1+\cdots+y_n$ and $x\varepsilon_i=y\varepsilon_i$. Then $x_i=y_i$. Now $x'=x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_n$ and $y'=y_1+\cdots+y_{i-1}+y_{i+1}+\cdots+y_n$ belong to $\ker\varepsilon_i$. Also $y+x'\in y+\ker\varepsilon_i$ and $x+y'\in x+\ker\varepsilon_i$. But $y+x'=x+y'$ since $x_i=y_i$.

Consequently $x+\ker\varepsilon_i \cap y+\ker\varepsilon_i \neq \phi$ and ε_i is semimaximal. Since $\ker\varepsilon_i = R_1 + \cdots + R_{i-1} + R_{i+1} + \cdots + R_n$ and $R\varepsilon_i = R_i$, it follows that $R = R\varepsilon_i + \ker\varepsilon_i$ and ε_i is complete. Thus ε_i is idempotent, semimaximal and complete. An endomorphism of a semiring that is idempotent, semimaximal and complete will be called a projection. It is straightforward to show that a projection on a semiring determines an internal direct sum.

5. Decomposable semirings

DEFINITION 5.1. A semiring is *decomposable* if $R = R_1 \oplus R_2$ and each R_i is a proper ideal in R. If R is not decomposable, then R is called *indecomposable.*

THEOREM 5.2. *A semiring R is decomposable if and only if there are projections of R that are not 0_R and not 1_R.*

PROOF. Suppose $R = R_1 \oplus R_2$ is decomposable. Then $R_i \neq 0$ and $R_i \neq R$, and it follows that the projections $\varepsilon_i \neq 1_R$ and $\varepsilon_i \neq 0_R$. Conversely, suppose ε is a projection of R such that $\varepsilon \neq 1_R$ and $\varepsilon \neq 0_R$. Let $R\varepsilon = R_1$ and $\ker\varepsilon = R_2$. Since ε is a projection, $R = R\varepsilon + \ker\varepsilon = R_1 + R_2$. It is clear that R_1 and R_2 are ideals in R. Now suppose that $x \in R_1 \cap R_2$. Then $x = y\varepsilon$ for some $y \in R$ and $x\varepsilon = 0$. Hence $x = y\varepsilon = y\varepsilon^2 = x\varepsilon = 0$ and $R_1 \cap R_2 = 0$. If x_1 and $y_1 \in R_1$, x_2 and $y_2 \in R_2$, and $x_1 + x_2 = y_1 + y_2$, then $(x_1 + x_2)\varepsilon = (y_1 + y_2)\varepsilon$. But $x_2\varepsilon = 0 = y_2\varepsilon$ and it follows that $x_1\varepsilon = y_1\varepsilon$. Consequently, $x_1 = y_1$ and $R_1 \cap R_2 = 0$ gives that R_1 and R_2 are related. Therefore theorem 3.1 assures that $R = R_1 \oplus R_2$.

DEFINITION 5.3. A semiring R satisfies *the descending chain condition* if $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ is a decreasing sequence of ideals in R, then there is an integer N such that $R_N = R_{N+1} = \cdots$

It is easy to show that any non-trivial semiring that satisfies the descending chain condition can be expressed as a direct sum of a finite number of indecomposable semirings.

6. The Krull-Schmidt theorem

A uniqueness theorem will now be given for direct sums of indecomposable semirings.

DEFINITION 6.1. A semiring R is said to satisfy *the ascending chain condition* if $R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$ is an ascending sequence of ideals in R, then there is an integer N such that $R_N = R_{N+1} = \cdots$

DEFINITION 6.2. An endomorphism η of a semiring R is called ***normal*** if the image of an ideal under η is an ideal.

LEMMA 6.3. If η: $R \rightarrow S$ is a semimaxal semiring homomorphism and $\ker\eta = \{0\}$, then η is one-to-one.

PROOF: Suppose $a\eta = b\eta$. Since η is semimaximal it follows that $a+\ker\eta \cap b+\ker\eta \neq \phi$. But $\ker\eta = \{0\}$. Consequently, $a+\ker\eta = b+\ker\eta$ and $a=b$. Therefore η is one-to-one.

THEOREM 6.4. *Let R be a semiring that satisfies both chain conditions. If η is a normal semimaxial endomorphism of R such that η is one to one or onto, then η is an automorphism.*

PROOF. Assume that η is one to one. If $R\eta^{t-1} = R\eta^{t}$ for some t, any $y \in R\eta^{t-2}$ is such that $y\eta = x\eta^{t} = (x\eta^{t-1})\eta$ for some x. Consequently, since η is one-to-one, $y = x\eta^{t-1} \in R\eta^{t-1}$ and $R\eta^{t-2} = R\eta^{t-1}$. Continuing in this manner we obtain $R = R\eta$. Consequently, if $R \supset R\eta$ and η is normal, then $R \supset R\eta \supset R\eta^{2} \supset \cdots$ is an infinite proper decreasing sequence of ideals in R. A contradiction, since R satisfies the descending chain condition. Therefore if η is one-to-one, $R = R\eta$ and η is an automorphism. Now suppose that $R\eta = R$ and $K_i = \ker\eta^{i}$ for $i = 0, 1, 2, \cdots$. Letting $\eta^{0} = 0$ gives $K_0 = \{0\}$. It is easy to see that $K_{t-1} \subset K_t$ and each K_t is an ideal in R. Suppose $K_{r-1} = K_r$ for some r and $z \in K_{r-1}$. Write $z = y\eta$. Then $0 = z\eta^{r-1} = (y\eta)\eta^{r-1} = y\eta^{r}$. Hence $y\eta^{r-1} = 0$ and $z\eta^{r-2} = (y\eta)\eta^{r-2} = y\eta^{r-1} = 0$. Consequently, $z \in K_{r-2}$ and it follows that $K_{r-2} = K_{r-1}$. Continuing in this manner one obtains $\{0\} = K_0 = K_1 = K_2 = \cdots$ Hence either $K_0 = \{0\}$ or $K_0 \subset K_1 \subset K_1 \subset K_2 \subset \cdots$ is an ascending sequence of ideals in R. Since R satisfies the ascending chain condition we must have $K_0 = \{0\}$. Since η is semimaximal it follows that η is one to one and hence an automorphism.

DEFINITION 6.5. If η is an endomorphism of a semiring, the set of elements z such that $z\eta^{t} = 0$ for some integer t is called the radical of η. It is clear that the radical of η is the union of all $\ker\eta^{i}$, $i = 0, 1, 2, \cdots$

THEOREM 6.6 (*Fittings lemma*). *Let R be a semiring that satisfies both chain conditions and η a normal complete endomorphism of R. Then $R = H \oplus K$ where K is the radical of η and $H = H\eta$.*

PROOF. Let $K_i = \ker\eta^{i}$ and consider ascending chain $K_0 \subset K_1 \subset \cdots$ and the descending chain $R \supset R\eta \supset R\eta^{2} \supset \cdots$. Since η is normal, each chain is a chain of ideals in R. By the chain conditions on R, there are integers r and s such that $R\eta^r = R\eta^{r+1} = \cdots$ and $K_s = K_{s+1} = \cdots$. Let $H = G\eta^r$ and $K = K_s$. Since η is complete, there is an integer t such that $R = R\eta^t + \ker\eta^t$. Let $p = \max\{r, s, t\}$. It follows that $R = R\eta^p + \ker\eta^p = H + K$ and K is the radical of η. Now let $\omega \in H \cap K$. Then $\omega = z\eta^p$ for suitable $z \in R$ and $\omega\eta^p = 0$. But $0 = \omega\eta^p = z\eta^{2p}$ and it follows that $z \in K$. Consequently, $0 = z\eta^p = \omega$ and $H \cap K = 0$. Suppose $x_1, x_2 \in H$, $y_1, y_2 \in K$ *and* $x_1 + y_1 = x_2 + y_2$. Then $(x_1 + y_1)\eta^p = (x_2 + y_2)\eta^p$. Thus $x_1\eta^p + y_1\eta^p = x_2\eta^p + y_2\eta^p$ and $x_1\eta^p = x_2\eta^p$ since $y_1, y_2 \in K$. But $H = R\eta^r = R\eta^{r+p} = (R\eta^r)\eta^p = H\eta^p$ and it follows from theorem 6.4 that η^p is an automorphism. Therefore η^p is one to one and $x_1 = x_2$. Consequently, H is related to K since $H \cap K = 0$. Therefore it follows from theorem 3.1 that $R = H \oplus K$.

COROLLARY 6.7. *If R is an indecomposable semiring that satisfies both chain conditions, then any normal complete endomorphism of R is either nilpotent or an automorphism.*

PROOF. From theorem 6.6, $R = H \oplus K$ where $H = H\eta$ and K is the radical of η. If R is indecomposable, either $R = H$ or $R = K$. If $R = H = H\eta$, then theorem 6.4 assures that η is an automorphism. If $R = K$, then η is nilpotent.

COROLLARY 6.8. *Let R be an indecomposable semiring that satisfies both chain conditions and η_1 and η_2 be normal complete endomorphisms. If $\eta_1 + \eta_2$ is an endomorphism, then $\eta_1 + \eta_2$ is nilpotent.*

The proof of corollary 6.8 is identical to the one for rings or groups and is omitted here.

THEOREM 6.9. (Krull-Schmidt theorem) *Let R be a semiring that satisfies both chain conditions and*

(1) $$R = H_1 \oplus \cdots \oplus H_s$$

(2) $$R = K_1 \oplus \cdots \oplus K_t$$

be two decompositions of R into indecomposable semirings. Then $s = t$ and for a suitable ordering of the K_i, we have $H_i \cong K_i$ and

(3) $$R = K_1 \oplus \cdots \oplus K_p \oplus H_{p+1} \oplus \cdots \oplus H_s.$$

PROOF. Suppose we have $K_1, K_2, \cdots, K_{r-1}$ paired with $H_1, H_2, \cdots, H_{r-1}$ in such a way that $K_i \cong H_i$ for $1 \leq i \leq r-1$ and (3) holds for $p \leq r-1$. Consider

(4) $$R = K_1 \oplus \cdots \oplus K_{r-1} \oplus H_r \oplus \cdots \oplus H_s.$$

Let $\lambda_1, \cdots \lambda_s$ be the projections determined by (4) and $\eta_1, \eta_2, \cdots \eta_t$ be the projections determined by (2). Now $\lambda_r=(\sum_1^t \eta_j)\lambda_r=\sum_1^t \eta_i\lambda_r$. For $x\in R$, $x\eta_j\in K_j$ and $j\leq r-1$ we have $x\eta_j=x\eta_j\lambda_j$ and $x\eta_j\lambda_r=x\eta_j\lambda_j\lambda_r=0$. Thus $\eta_j\lambda_r=0_R$ and $\lambda_r=\eta_r\lambda_r+\eta_{r+1}\lambda_r+\cdots+\eta_t\lambda_r$. In H_r, $\lambda_r=1_R$ and hence $\sum_r^t \eta_j\lambda_r=1_R$. Also any partial sum $\sum \eta_{ik}\lambda_r=(\sum\eta_{ik})\lambda_r$ induces a normal endomorphism in H_r. Since H_r is indecomposable it follows from corollary 6.8 that there exists a u, $r\leq u\leq t$ such that $r_u\lambda_r$ defines an automorphism of H_r. We can renumber the K_i such that K_u becomes K_r. Then $\eta_r\lambda_r$ is an automorphism of H_r. Let θ be its inverse. Then $H_r\xrightarrow{\theta}H_r\xrightarrow{\eta_r}K_r\xrightarrow{\lambda_r}H_r$ is the identity on H_r. Let $\alpha=\theta\eta_r\lambda_r=1_{H_r}$. Consider the normal composite β: $K_r\to K_r$ defined by $K_r\xrightarrow{\lambda_r}H_r\xrightarrow{\theta_r}H_r\xrightarrow{\eta_r}K_r$. Since $\alpha=1_{H_r}$, it follows that $\beta\beta=\beta$ and β is idempotent. Since K_r is indecomposable with both chain conditions, corollary 6.7 assures that β is nilpotent or β is an automorphism. Hence either $\beta=0_{K_r}$ or $\beta=1_{K_r}$. But $\beta\neq 0_{K_r}$ since β occurs in the composite $\alpha\alpha=1_{H_r}$. Therefore $\beta=1_{K_r}$ and η_r: $H_r\to K_r$ is an isomorphism. Now λ_r sends each element of $K_1+\cdots+K_{r-1}+H_{r+1}+\cdots+H_s$ onto 0 and since λ_r induces an isomorphism of K_r, $K_r\cap(K_1+\cdots+K_{r-1}+H_{r+1}+\cdots+H_s)=0$.

Now let $\overline{R}=K_1+\cdots+K_r+H_{r+1}+\cdots+H_s$. Then

$$\overline{R}=K_1\oplus\cdots\oplus K_r\oplus H_{r+1}\oplus\cdots\oplus H_s.$$

To show that $\overline{R}\simeq R$. If $x=x_1+\cdots+x_s$, $x_i\in K_i$ for $i\leq r-1$ and $x_j\in H_j$ for $j\geq r$, then ϕ: $x_1+\cdots+x_s\to x_1+\cdots+x_{r-1}+x\eta_r+x_{r+1}+\cdots+x_s$ is a normal endomorphism of R. Since η_r is an isomorphism, ϕ is an isomorphism of R onto $\overline{R}$. It follows from theorem 6.4 that $R=\overline{R}$. Thus (3) holds also for $r=s$ and the proof is complete.

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