Kyungpook Math. J.
Volume 17, Number 2
December, 1977.

ON *P*-COMPACT ORDERED SPACES

By Y. S. Park

0. Introduction

In [1], the concepts of I-compact, R-compact ordered spaces are introduced and they showed that the categories of I-compact and R-compact ordered spaces are epireflective in the category of completely regular ordered topological spaces.

In this note, using the concept of a P-compact ordered space (see [2]) and introducing the concept of a P-completely regular ordered space, we show that the category *PCOS* of P-compact ordered spaces is epireflective in the category *PCROS* of P-completely regular ordered spaces, which generalize the above mentioned result in [1].

For general categorical background and terminology we refer to [5] and for partially ordered topological spaces to [6, 7]

1. P-completely regular ordered spaces

By a partially ordered topological space $(X, \leq, \tau)$ we mean a set X endowed with both a partial order $\leq$ and a topology τ. Let $(X, \leq, \tau)$ be a partially ordered topological space. Then the partial order $\leq$ is called continuous provided that whenever $x \not\leq y$ in X, there are open sets U and V, $x \in U$ and $y \in V$, such that if $u \in U$ and $v \in V$, then $u \not\leq v$.

Let *POTS* be the category of partially ordered topological spaces and continuous increasing maps, and let *HOTS* be the category of partially ordered topological spaces with continuous orders and continuous increasing maps. It is known [4] that *POTS*, *HOTS* are complete.

DEFINITION 1.1. Let P be an object of *POTS*. An object X of *POTS* is called *P-completely regular ordered* if X is iseomorphic with a subspace of a power of P.

The category of P-completely regular ordered spaces and continuous increasing maps will be denoted by *PCROS*. We note that *PCROS* is a hereditary, productive category.

THEOREM 1.2. *PCROS is an epireflective subcategory of POTS.*

PROOF. Given $X \in POTS$, let $C_1(X, P)$ denote the family of continuous increasing maps from X into P. Define $\varphi : X \to P^{|C_1(X,P)|}$ by $\varphi(x)(f) = f(x)$ for each $f \in C_1(X, P)$ and each $x \in X$, where $|C_1(X, P)|$ denotes the cardinal number of $C_1(X, P)$. Then φ is a continuous increasing map. Then $\varphi(X)$ is a P-completely regular ordered space with the relative topology and the induced order from $P^{|C_1(X,P)|}$, that is, $\varphi(X) \in PCROS$. Now, we show that for any $Y \in PCROS$ and for any $POTS$-morphism $f: X \to Y$, there exists a unique $PCROS$-morphism $\bar{f}: \varphi(X) \to Y$ such that $\bar{f} \circ \varphi = f$. In fact, we indentify Y with a subspace of $P^{|S|}$, where S is a set, since Y is a P-completely regular ordered space. For every $s \in S$, let $f_s = p_s \circ f$, where p_s is the s th projection from $P^{|S|}$ onto P. It is easy to show that there exists a unique continuous increasing map $\bar{f}_s : \varphi(X) \to P$ such that $\bar{f}_s \circ \varphi = f_s$ for each $s \in S$. Define $\bar{f} : \varphi(X) \to P^{|S|}$ by $\bar{f}(y) = (\bar{f}_s(y))_{s \in S}$ for each $y \in \varphi(X)$. Then $\bar{f}$ is a continuous increasing map from $\varphi(X)$ into Y. Moreover, the uniqueness of $\bar{f}$ is immediate from the surjectivity of φ.

REMARK. We note that if $P \in HOTS$, then $PCROS$ is an epireflective subcategory of $HOTS$. Let $\boldsymbol{I}(\boldsymbol{R})$ denote the unit interval (the real line) endowed with the usual topology and the usual order. Taking $P = \boldsymbol{I}$(or $\boldsymbol{R}$), $CrORR$ is an epireflective subcategory of $POTS$ which generalizes a result in [1], where $CrORR$ is the category of completely regular ordered spaces and continuous increasing maps.

The proof of the following proposition is similar to the case of topological spaces and we will omit it.

PROPOSITION 1.3 *Let X be an object of POTS. Then X is P-completely regular ordered if and only if the following conditions hold.*

(1). *For every x, y in X with $x \not\leq y$, there exists a continuous increasing map $f: X \to P$ such that $f(x) \not\leq f(y)$.*

(2). *For every closed subset A of X and every point $x \in X - A$, there exists a positive integer n and a continuous increasing map $f: X \to P^n$ with $f(x) \notin \overline{f(A)}$, where $\overline{f(A)}$ denotes the closure $f(A)$ in P^n.*

2. *P*-compact ordered spaces

DEFINITION 2.1 ([2]). Let P be an object of $POTS$. An object X of $POTS$ is called *P-compact ordered* if X is iseomorphic with a closed subspace of a power of P.

The category of P-compact ordered spaces and continuous increasing maps will be denoted by *PCOS*. We note that *PCOS* is a closed hereditary, productive category.

THEOREM 2.2. *Let P be an object of HOTS. Then PCOS is an epireflective subcategory of PCROS.*

PROOF. For given $X \in PCROS$, the map $\sigma : X \to P^{|C_1(X, P)|}$ defined by $\sigma(x) = (f(x))_{f \in C_1(X,P)}$ is clearly an iseomorphism of X into $P^{|C_1(X,P)|}$. Let $\beta_{0P}X = \overline{\sigma(X)}$ and let $\overline{\sigma(X)}$ have the relative topology and the induced order from $P^{|C_1(X,P)|}$. By the same arguments as those in Theorem 1.2, there exists a unique continuous increasing map $\bar{f} : \beta_{0P}X \to P$ such that $\bar{f} | X = f$ for every $f \in C_1(X, P)$. Let Y be any object of *PCOS* and let $g : X \to Y$ be any continuous increasing map. We show that there exists a unique continuous increasing map $h : \beta_{0P} X \to Y$ such that $h | X = g$. In fact, since Y is an P-compact ordered space, Y is iseomorphic to a closed subspace of $\Pi \{P_\alpha : \alpha \in \Gamma\}$, where $P_\alpha = P$ for every $\alpha \in \Gamma$. For each α th projection map p_α, put $g_\alpha = p_\alpha \circ g$. Then g_α is a continuous increasing map from X into P, and hence there exists a unique continuous increasing map $\bar{g}_\alpha : \beta_{0P}X \longrightarrow P$ such that $\bar{g}_\alpha | X = g_\alpha$. Now, define $h : \beta_{0P} X \longrightarrow \Pi P_\alpha$ by $h(x)(\bar{g}_\alpha) = \bar{g}_\alpha(x)$ for each $x \in \beta_{0P} X$ and for each $\bar{g}_\alpha$, $\alpha \in \Gamma$. Then h is a continuous increasing map. If $x \in X$, then $\bar{g}_\alpha(x) = g_\alpha(x) = p_\alpha \circ g(x)$. Hence $h(x) = g(x)$ for all $x \in X$. Finally, since X is dense in $\beta_{0P} X$, $h(X)$ is dense in $h(\beta_{0P}X)$. But, Y is closed in ΠP_α, and hence $h(X) \subset Y$. It follows that $h(\beta_{0P} X) \subset Y$. Therefore, h is the required extension of g.

REMARKS 1. Let P be an object of *HOTS*. Then we obtain that *PCOS* is an epireflective subcategory of *HOTS*.

2. We observe that if Y is a P-compact ordered space containg X densely and inducing the order of X, where $P \in HOTS$, such that every continuous increasing map $f : X \longrightarrow P$ has a continuous increasing extension to Y, then Y is iseomorphic with $\beta_{0P} X$. We call $\beta_{0P} X$ in the above theorem the P-ordered compactification of X.

COROLLARY 2.3 ([1]). *Let IcOT and RcOT be the categories of I-compact ordered and R-compact ordered spaces with continuous increasing maps, respectively. Then IcOT and RcOT are both epireflective subcategories of CrORR.*

PROOF. Taking $P = I$(or$R$), $PCOS = IcOT$(or $RcOT$) and $PCROS = CrORR$. Hence the proof follows immediately.

COROLLARY 2.4. *Let P be a object of HOTS, and let X be an P-completely regular ordered space. Then X is P-compact ordered if and only if* $X=\beta_{0P}X$.

Kyungpook University

REFERENCES

[1] Choe, T.H. and Garcia, O.C., *Epireflective subcategories of partially ordered topological spaces,* Kyungpook Math. J. 13. (1973) 97-107.

[2] ______, and Hong, Y.H., *Extensions of completely regular ordered spaces,* Pacific J. Math. 66(1976) 37-48.

[3] Engelking, R., *Outline of General Topology,* John Wiley & Sons, New York, 1968.

[4] Garcia, O.C., *Partially ordered topological spaces,* Doct. Dissertation, McMaster University, Hamilton, Ont., 1972.

[5] Herrlich, H and strecker, G.E., *Category Theory,* Allyn and Bacon Inc., Boston, 1973.

[6] Nachbin, L., *Topology and Order,* Van Nostrand Mathematical Studies 4, Princeton, N.J., 1965.

[7] Ward Jr., L.E., *Partially ordered topological spaces,* Proc. Amer. Math. Soc. 5(1954) 144-161.