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A TIME OPTIMAL CONTROL PROBLEM

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In this note we consider a linear non-autonomous control system in the Euclidean n-space R^n, $\frac{dx}{dt}=A(t)x(t)+B(t)u(t)$, $x(t_0)=x_0$, equipped with the $L_{r,p}$-norm bounded control function over finite intervals of $[0,\infty)$. A necessary and sufficient condition for the controllability, the existence of the time optimal control function, the form of the optimal control function and related results are given by means of functional analysis methods.

1. Preliminaries.

Consider a linear non-autonomous control system in R^n

$$(1.1)\qquad \frac{dx}{dt}=A(t)x(t)+B(t)u(t),\quad x(t_0)=x_0$$

where $x(t)\in R^n$, $u(t)\in R^m$, A and B are $n\times n$-and $n\times m$-matrix valued functions respectively, whose components are Lebesque integrable on finite intervals of $[0,\infty)$.

The solution of the equation (1.1) has the form

$$(1.2)\qquad x(t)=X(t,t_0)x_0+X(t,t_0)\int_{t_0}^{t}X^{-1}(\tau,t_0)B(\tau)u(\tau)d\tau$$

where $X(t,t_0)$ is the fundamental matrix of the homogeneous linear differential equation corresponding to the equation (1.1), that is,

$\frac{d}{dt}X(t,t_0)=A(t)X(t,t_0)$, $X(t_0,t_0)=I)$ (I=the identity matrix). The solution of (1.1) is unique if the initial condition $x(t_0)=x_0$ and the control funtrol function u are fixed. We will denote $x(t)=x(t\,;t_0,\ x_0,\ u)$

In this note, we will equip a control u with a norm over a finite interval $K=[t_0,t_1]$ which will be $L_{r,p}$-norm, that is, for $u\in L_{r,p}$

$$\|u\|_{r,p}=(\int_{t_0}^{t_1}\|u(\tau)\|^p{}_r d\tau)^{1/p},\ 1\leq p<\infty,$$

where
$$\|u(\tau)\|_r=(\sum_{i=1}^{m}|u_i(\tau)|^r)^{1/r},\quad 1\leq r<\infty.$$

Let $U_{r,p}^{\rho}$ be the class of admissible controls with $\|u\|_{r,p} \leq \rho$; that is,

$$U_{r,p}^{\rho} = \{u \in L_{r,p} : \|u\|_{r,p} \leq \rho < \infty\}.$$

TIME OPTIMAL CONTROL PROBLEM. Given an initial state x_0 at time t_0, we wish to transfer it by means of a control function $u \in U_{r,p}^{\rho}$ to a target point x_1, and we want to do this in the shortest time possible, that is, we seek an $u_* \in U_{r,p}^{\rho}$ and a time t_* such that $x(t; t_0, x_0, u_*) = x_1$, $x_1 \in R^n$, while $x(t; t_0, x_0, u) \neq x_1$ for each $u \in U_{r,p}^{\rho}$ if $t < t_*$.

For the above optimal problem, we will discuss the following existence problem:

"Does an $u \in U_{r,p}^{\rho}$ exist such that $x(t; t_0, x_0, u) = x_1$?", or "Is the system (1.1) controllable with respect to x_1, $U_{r,p}^{\rho}$, K?"

We will give an answer to this question in the next section.

2. Controllability and Time optimal controls.

To solve the existence problem, we define the linear operator $T: U_{r,p}^{\rho} \to R^n$ by

$$Tu = \int_{t_0}^{t_1} X(t_1, t_0) X^{-1}(\tau, t_0) B(\tau) u(\tau) d\tau.$$

If we put $M(t_1, \tau) = X(t_1, t_0) X^{-1}(\tau, t_0) B(\tau)$, then

$$Tu = \int_{t_0}^{t_1} M(t_1, \tau) u(\tau) d\tau.$$

We also define the reachable set

$$\boldsymbol{R}(t) = \{x(t; t_0, x_0, u) : x(t; t_0, x_0, u) = X(t, t_0) x_0 + Tu, \ u \in U_{r,p}^{\rho}\}.$$

It is known that $\boldsymbol{R}(t)$ is compact and convex in R^n which varies continuously with $t > 0$ with respect to Hausdorff metric

$$h(\boldsymbol{R}(t), \boldsymbol{R}(s)) = \max\{\sup d(a, \boldsymbol{R}(s)), \sup d(\boldsymbol{R}(t), b)\}, \ a \in \boldsymbol{R}(t), b \in \boldsymbol{R}(s)\}$$

The following lemma is also known:

LEMMA 1. *Two convex compact sets A and B in R^n have a nonempty intersection if and only if*

$$\min\{\langle x, b\rangle : b \in B\} \leq \max\{\langle x, a\rangle : a \in A\} \quad \text{for any } x \in R^n.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the scalar product in R^n defined by

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i, \quad x = (x_1, x_2, \cdots, x_n), \quad y = (y_1, y_2, \cdots, y_n).$$

A norm of $x \in R^n$ will be defined by $\|x\| = \langle x, x\rangle^{1/2}$.

Using Lemma 1, we will prove the following

THEOREM 1. Assume $Tu=x_1-X(t_1,t_0)x_0$, for some $u\in U_r{}^{\rho},{}_p$, *the system* $dx/dt=A(t)x+B(t)u$, $x(t_0)=x_0$, *is controllable with respect to* x_1, $U_r{}^{\rho},{}_p$, $K=[t_0,t_1]$ $(0\leq t_0<t_1)$ *if and only if*

$$|\langle x, x_1-X(t_1,t_0)x_0\rangle|\leq\rho\sqrt{n}\,\|x\|\|M(t_1)\|_{s,q} \text{ for any } x\in R^n,$$

where $\frac{1}{r}+\frac{1}{s}=1$, $\frac{1}{p}+\frac{1}{q}=1$, $\|M(t_1)\|_{s,q}=(\int_{t_0}^{t_1}\|M(t_1,\tau)\|_s{}^q d\tau)^{1/q}$,

$\|M(t_1,\tau)\|_s=\sup_{1\leq i\leq n}(\sum_{j=1}^{m}|(M(t_1,\tau))_{i,j}|^s)^{1/s}$.

Proof. By Lemma 1 if we put $B=\boldsymbol{R}(t)$ and $A=\{x_1\}$, we see that the system is controllable with respect to x_1, $U_r{}^{\rho},{}_p$, $K=[t_0,t_1]$ if and only if

$$\min[\{\langle x,y\rangle : y\in\boldsymbol{R}(t)\}]\leq\langle x,x_1\rangle, \text{ for any } x\in R^n$$

or

$$\min[\{\langle x, X(t_1,t_0)x_0\rangle+\langle x, Tu\rangle\} : u\in U_r{}^{\rho},{}_p]\leq\langle x_0, x_1\rangle.$$

Thus we have

(2.0) $\langle x, X(t_1,t_0)x_0\rangle-\max[\{\langle x, Tu\rangle\} : u\in U_r{}^{\rho},{}_p]\leq\langle x_0,x_1\rangle$, for any $x\in R^n$.

Since x is arbitrary,

(2.1) $$|\langle x, x_1-X(t_1,t_0)x_0\rangle|\leq\max_u|\langle x, Tu\rangle|.$$

Moreover, since $Tu=\int_{t_0}^{t_1}M(t_1,\tau)u(\tau)d\tau$ and $M(t_1,\tau)$ is an $n\times m$-matrix, if we put

$$M(t_1,\tau)=\left(a_{ij}(t_1,\tau)\right),\quad i=1,2,\cdots,n;\ j=1,2,\cdots,m, \text{ that is,}$$

$$\left(M(t_1,\tau)\right)_{i,j}=a_{ij}(t_1,\tau) \text{ and } u(\tau)=\mathrm{col}(u_1(\tau),\cdots,u_m(\tau)),$$

then we have

$$\int_{t_0}^{t_1}M(t_1,\tau)u(\tau)d\tau=(\int_{t_0}^{t_1}\sum_{j=1}^{m}a_{1j}(t_1,\tau)u_j(\tau)d\tau,\cdots,\int_{t_0}^{t_1}\sum_{j=1}^{m}a_{nj}(t_1,\tau)u_j(\tau)d\tau).$$

Hence

$$\left\|\int_{t_0}^{t_1}M(t_1,\tau)u(\tau)d\tau\right\|_{R^n}=\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\int_{t_0}^{t_1}a_{ij}(t_1,\tau)u_j(\tau)d\tau\right)^2\right]^{1/2}$$

$$\leq\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{m}\int_{t_0}^{t_1}|a_{ij}(t_1,\tau)u_j(\tau)|d\tau\right)^2\right]^{1/2}$$

By the Hölder inequality, it follows that

$$(2.2)\qquad \left\|\int_{t_0}^{t_1} M(t_1,\tau)u(\tau)\,d\tau\right\|_{R^n}^2 \le \sum_{i=1}^{n}\left(\int_{t_0}^{t_1}\sum_{j=1}^{m}|a_{ij}(t_1,\tau)|\,|u_j(\tau)|\,d\tau\right)^2$$

$$\le \sum_{i=1}^{n}\left[\int_{t_0}^{t_1}\left(\sum_{j=1}^{m}|a_{ij}(t_1,\tau)|^s\right)^{1/s}\left(\sum_{j=1}^{m}|u_j(\tau)|^r\right)^{1/r}d\tau\right]^2$$

where $\frac{1}{r}+\frac{1}{s}=1,\ 1<r<\infty.$

Since $\left(\sum_{j=1}^{m}|u_j(\tau)|^r\right)^{1/r}=\|u(\tau)\|_r$ and

$$\sup_{1\le i\le n}\left(\sum_{j=1}^{m}|a_{ij}(t_1,\tau)|^s\right)^{1/s}=\|M(t_1,\tau)\|_s,$$

applying the Hölder inequality once again, we have

$$\left\|\int_{t_0}^{t_1} M(t_1,\tau)u(\tau)\,d\tau\right\|_{R^n}^2 \le n\left(\int_{t_0}^{t_1}\|M(t_1,\tau)\|_s\|u(\tau)\|_r d\tau\right)^2$$

$$(2.3)\qquad \le n\left[\left(\int_{t_0}^{t_1}\|M(t_1,\tau)\|_s^q d\tau\right)^{1/q}\left(\int_{t_0}^{t_1}\|u(\tau)\|_r^p d\tau\right)^{1/p}\right]^2.$$

It follows from (2.1) and (2.3) that

$$|\langle x, x_1-X(t_1,t_0)x_0\rangle| \le \sqrt{n}\,\|x\|_{R^n}\|M(t_1)\|_{s,q}\ \max_u\|_{r,p}$$

Since $u\in U_r{}^{\rho}{}_{,p}$, we have proved the necessary condition.

Conversely, suppose that the condition of the Theorem 1 holds. Since

$$|\langle x, Tu\rangle| \le \rho\sqrt{n}\,\|x\|\|M(t_1)\|_{s,q}$$

holds automatically, we have

$$\langle x, x_1-X(t_1,t_0)x_0\rangle=\langle x, Tu\rangle\le\max_u\langle x, Tu\rangle$$

$$\le\max_u|\langle x, Tu\rangle|\le\rho\sqrt{n}\,\|x\|\|M(t_1)\|_{s,q}.$$

Thus from the Lemma 1 and (2.0) it follows that the system is controllable. This completes the proof. Q. E. D.

If we replace x by Tu in Theorem 1, then we have

$$|\langle Tu, x_1-X(t_1,t_0)x_0\rangle|\le\rho\sqrt{n}\,\|Tu\|\|M(t_1)\|_{s,q},\quad u\in U_r{}^{\rho}{}_{,p}.$$

Since

$$Tu=x_1-X(t_1,t_0)x_0 \text{ for some } u\in U_r{}^{\rho}{}_{,p}.$$

$$\|Tu\|\le\rho\sqrt{n}\,\|M(t_1)\|_{s,q}.$$

Thus we have

$$\sup\{\|Tu\| : u \in U_r^\rho{}_{,p}\} = \sup\{\|Tu\| : \|u\|_{r,p} \le \rho\}$$
$$\le \rho\sqrt{n}\,\|M(t_1)\|_{s,q}.$$

This yields that

$$\rho \cdot \sup\{\|Tu\| : \|u\|_{r,p} \le 1\} = \rho\|T\| \le \rho\sqrt{n}\,\|M(t_1)\|_{s,q}.$$

Therefore we have the following

COROLLARY 1. *If the system is controllable with respect to* x_1, $U_r^\rho{}_{,p}$, $K=[t_0, t_1]$, *then*

$$\|T\| \le \sqrt{n}\,\|M(t_1)\|_{s,q},$$

where $Tu = \int_{t_0}^{t_1} M(t_1, \tau)u(\tau)\,d\tau$ *and* $\|M(t_1)\|_{s,q} = \left(\int_{t_0}^{t_1} \|M(t_1, \tau)\|_s^q d\tau\right)^{1/q}$.

Now we will prove the existence of the time optimal control fuction by using Theorem 1.

Under the same condition in Theorem 1, we have

THEOREM 2. *If a system* $\dfrac{dx}{dt} = A(t)x + B(t)u$, $x(t_0) = x_0$ *is controllable with respect to* x_1, $U_{r,p}$, $K=[t_0, t_1]$, *then there exists a time optimal control function.*

Proof. Consider the set $S=\{t \cdot\cdot t > t_0 \ge 0\}$ over which the system is controllable. Let $t_* = \inf S$. We must prove that the system is controllable with respect to x_1, $U_r^\rho{}_{,p}$ and $K^* = [t_0, t_*]$, that is, the K^* is the minimal time interval for which the system is controllable which is equivalent to $t_* \in S$.

Suppose that $t_* \overline{\in} S$. Then by Theorem 1 there exists an $x' \in R^n$ such that

$$|\langle x', \ x_1 - X(t_*, t_0)x_0\rangle| > \rho\sqrt{n}\,\|x'\|_{R^n}\|M(t_*)\|_{s,q}.$$

According to the definition of t_*, there exists a sequence $\{t_k\}$ with $t_k \to t_*$ such that for each k and $K_k = [t_0, t_k]$ the system is controllable. Therefore we have

$$|\langle x_k', x_1 - X(t_k, t_0)x_0\rangle| \le \rho\sqrt{n}\,\|x_k'\|\|M(t_k)\|.$$

Since $X(t_k, t_0)$ and the right hand side of the above inequality are continuous for t_k, it follows that

$$|\langle x', x_1 - X(t_*, t_0)x_0\rangle| \le \rho\sqrt{n}\,\|x'\|\|M(t_*)\|,$$

which is a contradiction. Q. E. D.

THEOREM 3. *If* $K^* = [t_0, t_*]$ *is the minimal time interval on which the system* $dx/dt = A(t)x + B(t)u$, $x(t_0) = x_0$ *is controllable with respect to* x_1 $U_r^\rho{}_{,p}$, K^*, *then there exists an* $x' \in R^n$ *such that*

$$(2.4)\qquad |\langle x', x_1 - X(t_*, t_0)x_0\rangle| = \rho\sqrt{n}\,\|x'\|_{R^n}\|M(t_*)\|_{s,q},$$

where $\|M(t_*)\|_{s,q} = \left(\int_{t_0}^{t_*}\|M(t_*, \tau)\|_s^q d\tau\right)^{1/q}$.

Proof. By the definition)of t_* and Theorem 1, if we take a sequence $\{t_k\}$ with $t_k < t_*$, $t_k \to t_*$, then there exists a sequence of vectors $\{x_k\}$ in R^n corresponding to $\{t^k\}$ with $x_k \to x'$, and

$$|\langle x_k, x_1 - X(t_k, t_0)x_0\rangle| > \rho\sqrt{n}\,\|x_k\|\|M(t_k)\|.$$

Hence we have

$$|\langle x', x_1 - X(t_*, t_0)x_0\rangle| \geq \rho\sqrt{n}\,\|x'\|\|M(t_*)\|.$$

On the other hand, the system is controllable with respect to x', $U_{r,p}^{\rho}$ and K^*. This implies that

$$|\langle x', x_1 - X(t_*, t_0)x_0\rangle| \leq \rho\sqrt{n}\,\|x'\|\|M(t_*)\|.$$

Thus we have the desired equality. Q. E. D.

Since the system is controllable with respect to x_1, $U_{r,p}^{\rho}$ and $K^* = [t_0, t_*]$, there exists a time optimal control $u^* \in U_{r,p}^{\rho}$ with minimal time interval K^*. Hence

$$\int_{t_0}^{t_*} M(t_*, \tau)u^*(\tau)\,d\tau = Tu^* = x_1 - X(t_*, t_0)x_0.$$

We have the following

COROLLARY 2. *Under the same condition as that in Theorem 3, we have*

$$\|Tu^*\| = \left\|\int_{t_0}^{t_*} M(t_*, \tau)u^*(\tau)\,d\tau\right\| = \rho\sqrt{n}\,\|M(t_*)\|_{s,q}.$$

Proof. From the equality (2.4), we see that

$$\rho\sqrt{n}\,\|x'\|\|M(t_*)\| \leq \|x'\|\|Tu^*\|, \text{ and hence } \rho\sqrt{n}\,\|M(t_*)\| \leq \|Tu^*\|.$$

On the other hand, the controllability implies that

$$|\langle Tu^*, Tu^*\rangle| \leq \rho\sqrt{n}\,\|Tu^*\|\|M(t_1)\|.$$

Hence $\qquad \|Tu^*\| \leq \rho\sqrt{n}\,\|M(t_1)\|$. Q. E. D.

Corollary 2 implies that the time optimal control function u^* makes the norm of $Tu = \int_{t_0}^{t_*} M(t_*, \tau)u(\tau)\,d\tau$, $u \in U_{r,p}^{\rho}$ maximum. Moreover, if $\rho = 1$ (that is, if we consider the class $U_{r,p}$), then Corollary 1 and Corollary 2 imply that $\|T\|=\|Tu^*\|=\sqrt{n}\|M(t_*)\|$. Therefore

$\|T\|=\max\{\|Tu\|:u\in U_{r,p}\}=\|Tu^*\|\leq\|T\|\|u^*\|=\|T\|$, hence $\|u^*\|_{r,p}=1$.

Thus we have the following result:

COROLLARY 3. *If $K^*=[t_0, t_1]$ is the minimal time interval on which the system $dx/dt=A(t)x+B(t)u$, $x(t_0)=x_0$ is controllable with respect to x_1, $U_{r,p}$ and K^*, then the maximum value $\max\left[\|\int_{t0}^{t_*}M(t_*,\tau)u(\tau)\,d\tau\|:u\in U_{r,p}\right]$ can be attained by an optimal control u^* with $\|u^*\|_{r,p}=1$, and*

$$\text{(2.5)}\quad \|T\|=\|Tu^*\|=\max\left[\|\int_{t_0}^{t_*}M(t_*,\tau)u(\tau)\,d\tau\|:u\in U_{r,q}\right]=\sqrt{n}\|M(t_*)\|_{s,q}.$$

Now we will deternine the form of the optimal contimal function u^* with the minimal time t_*. In the Hölder inequality

$$\left|\sum_{i=1}^{n}\alpha_i\beta_i\right|\leq\left(\sum_{i=1}^{n}\alpha_i^p\right)^{1/p}\left(\sum_{i=1}^{n}\beta_i^q\right)^{1/q},\quad \frac{1}{p}+\frac{1}{q}=1,$$

the equality holds if and only if $\alpha_i^p=k\beta_i^q$ for any positive constant k, and sign $\alpha_i=\text{sign}\beta_i$, $i=1,2,\cdots,n$, or equivalently,

$$\alpha_i=k^{1/p}|\beta_i|^{q/p}\ \text{sign}\beta_i.$$

From the inequality (2.2), (2.3) and Corollary 2, we have the following:

$$\begin{aligned}\|Tu^*\|^2=\|\int_{t_0}^{t_*}M(t_*,\tau)u^*(\tau)\,d\tau\|^2&\leq n\left(\int_{t_0}^{t_*}\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|\,|u_j^*(\tau)|\,d\tau\right)^2\\ &\leq n\left[\int_{t_0}^{t_*}\left(\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|^s\right)^{1/s}\left(\sum_{j=1}^{m}u^*_j(\tau)|^r\right)^{1/r}d\tau\right]^2\\ \text{(2.6)}\qquad &\leq n\left[\left(\int_{t_0}^{t_*}\|M(t_*,\tau)\|_s^q\,d\tau\right)^{1/q}\left(\int_{t_0}^{t_*}\|u(\tau)\|_r^p\,d\tau\right)^{1/p}\right]^2\\ &\leq\rho^2n\|M(t_*)\|_{s,q}{}^2=\|Tu^*\|^2,\end{aligned}$$

where $$\sum_j|a_{i_0j}(t_*,\tau)|=\max_{1\leq i\leq n}\sum_j|a_{ij}(t_*,\tau)|.$$

Thus we have

$$\int_{t_0}^{t_*}\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|\,|u_j^*(\tau)|\,d\tau=\int_{t_0}^{t_*}\left(\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|^s\right)^{1/s}\left(\sum_{j=1}^{m}|u_j^*(\tau)|^r\right)^{1/r}d\tau$$

and

$$u_j^*(\tau)=k^{1/r}|a_{i_0j}(t_*,\tau)|^{s/r}\ \text{sign}\ a_{i_0j}(t_*,\tau),\ j=1,2,\cdots,m$$

in the sense that almost everywhere on $[t_0, t_*]$.

It remains to determine the constant $k^{1/r}$, but since

$$\left(\sum_{j=1}^{m}|u_j^*(\tau)|^r\right)^{1/r}=\|u^*(\tau)\|_r,\quad \left(\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|^s\right)^{1/s}=\|M(t_*,\tau)\|_s,$$

it follows from (2.6) that

$$\left[\int_{t_0}^{t_*}\left(\sum_{j=1}^{m}k|a_{i_0j}(t_*,\tau)|^s\right)^{p/r}d\tau\right]^{1/p}=\rho.$$

Hence

$$k^{1/r}=\rho\Big/\left[\int_{t_0}^{t_*}\left(\sum_{j=1}^{m}|a_{i_0j}(t_*,\tau)|^s\right)^{p/r}d\tau\right]^{1/p}.$$

Therefore, a general form of the optimal control function u^* is given by

$$u_j^*(\tau)=\rho\Big/\left[\int_{t_0}^{t_*}\left(\sum_{j=1}^{m}|a_{j_0j}(t_*,\tau)|^s\right)^{p/r}d\tau\right]^{1/p}\cdot|a_{i_0j}(t_*,\tau)|^{s/r}\operatorname{sign} a_{i_0j}(t_*,\tau),$$

$j=1,2,\cdots,m$.

In the case when $r=s=2$, $p=q=2$, the optimal control function is given by

$$u_j^*(\tau)=\rho\Big/\left[\int_{t_0}^{t_*}\|M(t_*,\tau)\|_2^2d\tau\right]^{1/2}\cdot|a_{i_0j}(t_*,\tau)|\operatorname{sign} a_{i_0j}(t_*,\tau),$$

$j=1,2,\cdots,m$.

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