Bull. Korean Math. Soc.
Vol. 14, No. 1, 1977.

A NOTE ON THE KERNEL OF POINCARÊ OPERATOR

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A *Fucshian grop* G is a subgroup of the Mobious group, which acts discontinuously on the open unit disk $\varDelta$ of the Riemann sphere. The quotient $\varDelta/G$ is a 2-manifold and is given a complex structure by requiring that the projection $\varDelta \to \varDelta/G$ to be holomorphic. A holomorphic function $g(z)$ on $\varDelta$ satisfying the following functional equation

$g(A(z)) = A'(z)^q g(z)$ for $A \in G$, is called an *automorphic form* of *weight*-$2q$. Poincaré constructed an automorphic form by the following equation

$F(z) = \sum f(A(z)) A'(z)^q$; $f(z)$ is analytic on $\varDelta$ and $\sum$ moves on G. We denote it by $F(z) = (\theta_q)(f(z))$. A *fundamental region* w of a Fucshian group G is a connected $w \subset \varDelta$ such that mes $(\mathrm{cl}(w)/w)$ is zero, no two interior points of w are G equivalent, and every $z \in \varDelta$ is equivalent to some point of $\mathrm{cl}(w)$. The analytic automorphic form of weight-$2q$ with

$$\|f\|_q{}^p{}_{,p} = {}_w\lambda(z)^{2-qp} |f(z)|^p dxdy < \infty$$

form a Banach space $A_q(G)$ of *integrable forms*. The analytic function satisfying

$$\|f\|_{q,p} = \int_\varDelta \lambda(z)^{2-qp} |f(z)|^p dxdy < \infty$$

form a Banach space A_q.
It is well known that

$$\theta_q : A_q \to A_q(G)$$

is norm decreasing, surjective linear mapping. Let G be a Fucshian group. Are there any constructive characterizations of the kernel of the θ_q? This problem is an important and unknown. In the following we consider the above problem.

Let the Fuchsian group G contain an elliptic element E. Considering the conjugate group of it we may assume without loss of generality that the fixed points of E are 0 and ∞. E can be written as $E(z) = \alpha z$ and $\alpha = \exp(2\pi i/l)$. $\{E^n\}$ forms a subgroup of order 1.

PROPOSITION 1: *Let G be a Fucshian group contains an elliptic element of order l. $f(z)$ be an analytic function with $f(E(z)) = f(z)$. Then the kernel θ_q contains $f(z)$ if q is not a multiple of l.*

Proof. Consider the coset space $G/\{E\}$, where $\{E\}=\{E, E^2, \cdots, E^l\}$. Let H be a set which consists of exactly one elements from each coset of $G/\{E\}$. Then we have

$$G=\{E^n A : n=1, 2, \cdots, l, \quad A\in H\},$$
$$(d/dz)(\textstyle\sum_1^l E^i(z))=1.$$

Let $F(z)=\theta_q(f(z))$ then

$$F(z)=\textstyle\sum^* [\sum_1^l f(E_n(A(z))] E_n'(A(z))A'(z)^q,$$

where $E_n(z)$ denotes $E^n(z)$ and $\sum^*$ moves on H. By a simple calculation we have

$$F(z)=\textstyle\sum^* f(A(z))A'(z)^q[1-E'(z)^{lq}/1-E'(z)^q].$$

Hence if q is not a multiple of l then $E'(z)^l=1$ gives that $F(z)\equiv 0$.

COROLLARY. *G be a Fucskian group containing an elliptic element of order l with fixed points 0 and if $k+q$ and l are relative prime prime then $\theta_q(z_k)\equiv 0$.*

Proof. By a simple calculation we have

$$\theta_q(z^k)=\textstyle\sum^* A(z)^k A'(z)^q[1-E'(z)^{l(k+m)}/1-E'(z)^{k+m}]$$

Since $\{E\}$ is cyclic the asseetion follows.

Note the proposition 1 is proved in [4], in the upper half plane.

We assume now on that the Riemann surface Δ/G is compact. For analytic function $f(z)$ with $|f(z)|<\infty$ on Δ, it is clear that $f(z)\in A_q$ for all $p\geqslant 2$. For a compact Rimann surfacr $F(z)\in A_q(G)$ has only a finite number of zero's on Δ/G, otherwise it is identically zero. The number of zero points depends only on G. Let N be the number of zero's of $F\in A_q$ then we have the following formular

$$N=(q/2)\{2g-2+\textstyle\sum(1-l_i^{-1})\}$$

where g is the genus of Δ/G and l_i are constants depends on G. Let $\{z_1, z_2, \cdots, z_n\}$ be $n(>N)$ different points in w and if an analytic function ($|f|<\infty$), has zero's on the set

$$\{A(z_i) : i=1, 2, \cdots, n \text{ and } A\in G\}$$

then clearly $\theta_q(f(z))\equiv 0$. But a G *set* $X=\{A(z) : A\in G\}$ to be a zero set of a bounded analytic function on Δ, it is necessary that X satisfies the following

$$\textstyle\sum(1-|A(z)|)<\infty.$$

By a simple calculation we have

$$\sum(1-|A(z)|)>(1/2)(1-|z|^2)(\sum A'(z)).$$

For the nonelementary group G, it is known that $\sum|A'(z)|^q$ diverges for $p<1$. Hence we have the following proposition.

PROPOSITION 1. *Let G be non-elementary Fucshian group. A bounded analytic function has zero's on a G set then it vanishes identically on Δ.*

By definition $f\in A_q$ satisfies

$$\int_\Delta (1-|z|^2)^{q-2}|f(z)|dxdy<\infty.$$

Now we consider only unbounded f which is contained in A_q. With the Poincare metric $4\lambda(z)^{-1}$ we know that the area w depends only on G. Let l/kbe the area of w. Let $R(m, n)=\{z : 1-2^m<|z|<1-2^n\}$, then we have
Area $R(n+1,\ n)>2\pi(2^{n+1}-2^n)=(2\pi)(2^n)$. We call a Fucshian group G *proportional* if $R(n+1,\ n)$ contains at least $k2^n$ fundamental regions of G.

PROPOSITION 2. *Let G be a proportional group and $(q+1)<(\log_2 e)\ (k/4)$. Then analytic function f of A_q has zero's on a G set iff $f\equiv 0$.*

Proof. Let f has zero's on a G set $\{A(z) : A\in G\}$. Assume that f has no zero at $z=0$. Then by Jensen's formular we have

$$|f(0)|(\pi r/|A(z)|)<(1/2\pi)\int_0^{2\pi}|f(re^{i\theta})|d\theta$$

where π moves on all $|A(z)|<r$. Since $K(n, n+1)$ contains 2^nK zero points of f, we have the following estimation.

$$|f(0)|(\beta/\alpha)<(1/2\pi)^{2\pi}{}_0|f(re^{i\theta})|d\theta,$$

$$\log\ \alpha=\{K\log(1-(1/2))+2K\log(1-(1/2^2))+\cdots+2^{m-1}K$$
$$\log(1-(1/2^m))\}$$

$$\log\beta=(K+2K+\cdots+2^{m-1}K)\log(1-(1/2^m)).$$

$$\lim_{m\to\infty}\log\alpha<(-Km/4),$$

$$\lim_{m\to\infty}\log\beta\geq(-2K).$$

Hence

$$(|f(0)|)(\pi r/|A(z)|)\ \exp((Km/4)-2K)$$

as $m\to\infty$.
Let $t=(\log_2 e)(K/4)$, then $\exp((Km/2)-2K)$
diverges in the form of $(1-r)^t$.

Hence if $t>q+1$, then

$$\int_{\Delta}(1-|z|^2)^{q-2}|f(z)|\,dxdy$$

is diverging, and conclude the proposition.

Assume that G has no parabolic element, then the *bounded automorphic forms* (that is $|F(z)\lambda(z)^{-q}|<\infty$) with inner product

$$(F, G)=_w F(z)G(z)\lambda(z)^{2-q}dxdy$$

becomes a Hilbert space.

It is known that every bounded automorphic form of G has exactly N zero's on w, and the N depends only on G. Let G contain elliptic elements and l be the least order of elliptic subgroup of G. Further assume that the elliptic element has fixed points at 0 and ∞. Under this asumption a bounded automorphic form $F(z)$ can be expended as

$$F(z)=a_v z^v+\cdots+a_{v+nl}z^{v+nl}+\cdots,$$

where v is the smallest positive integer with $v+q$ is a multiple of l.

PROPOSITION 3. *Let G and v be defined as the adove. Then*

$$\{\theta(z^k) : v=k,\ k+1,\cdots,\ k+nl,\cdots\}$$

forms a basis of the Hilbert space of the of the bounded forms. And at least one of $\{z^k, z^{k+1},\cdots, z^{k+(N+1)l}\}$ *is not in the kernel of* θ_q.

Proof. By calculation we have $(F(z), \theta(z^k))=a_k c_k$ where a_k is the k-th taylor coefficient of $F(z)$. If $F(z)$ is orthogonal to all of $\theta(z^k)$, then we have $F(z)\equiv 0$. Hence $\{\theta(z^k)\}$ forms a basis. If $\theta(z^{k+il})\equiv 0$ for $i=0, 1,\cdots, N+1$. Then every bounded form have at least $N+1$ zero at the origin, it is a contradiction. Hence we conclude the proposition.

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