Bull. Korean Math. Soc.
Vol. 14, No. 1, 1977.

FIXED POINTS OF CONTRACTIVE MAPS OF COMPACT METRIC SPACES

BY SEHIE PARK

We consider various applications of Jungck's fixed point theorem [5] on compact metric spaces. In fact, we show that a number of known facts on fixed points are consequences of the theorem. Also applications to common fixed points of families of maps are considered.

1. Introduction

In [5], Jungck gave a very useful fixed point theorem on compact metic spaces. Namely,

THEOREM A. (Jungck [5]) *Let f be a continuous map of a compact metric space (X, d) into itself. Then f has a fixed point iff for any $x, y \in X$, $fx \neq fy$, there is a map g which commutes with f such that*

$$d(gx, gy) < d(fx, fy).$$

In this paper, we demonstrate the usefulness of Theorem A by listing its direct consequences. Also we discuss some mutual relations among known results. In fact, we list fixed point theorems on maps and later on common fixed points of families of self-maps of a compact metric space. Finally, we give some applications.

2. Preliminaries

A map f of a metric space (X, d) into itself is said to be *nonexpansive* if $d(fx, fy) \leqslant d(x, y)$ for all $x, y \in X$. f is said to be *contractive* if $d(fx, fy) < d(x, y)$ for any distinct points $x, y \in X$. f is said to be *iteratively contractive* if for any $x, y \in X$, $x \neq y$, there is an integer $k > 0$ such that $d(f^k(x), f^k(y)) < d(x, y)$. Let C_f denote the set of all maps $g : X \to X$ which commute with f. We note that if $z \in X$ is a fixed point of f, then so is $g^n(z)$ for any $g \in C_f$ and any integer $n > 0$.

Let G be a commutative semigroup of continuous self-maps of (X, d). Then G is said to be *proximally contractive* if for any $x, y \in X$, $x \neq y$, there is a $g \in G$ such that $d(gx, gy) < d(x, y)$. Two points $x, y \in X$ are said to be *ε-proximal* w. r. t. G if for any $\mu > 0$ and $f \in G$ with $d(fx, fy) < \varepsilon$ there exists $g \in G$ such that $d(gf(x), gf(y)) < \mu$. If x and y are ε-proximal for all $\varepsilon>0$, then they are said to be *proximal* [3].

The following is useful.

THEOREM B. (Holmes [3, Theorem 1]) *If (X, d) is compact and G is proximally contractive, then each pair of points in X is proximal.*

3. Fixed points of maps

Now we consider direct applications of Theorem A.

PROPOSITION 3.1. *Let f be a continuous self-map of a compact metric space (X, d). If for any $x, y\in X$, $fx\neq fy$, there is an integer $N=N(x, y)>0$ such that*

$$d(f^N x, f^N y)<d(fx, fy),$$

then f has a unique fixed point.

Proof. Since f^N belongs to C_f, f has a fixed point. The uniqueness follows from $d(f^N x, f^N y)<d(fx, fy))$.

PROPOSITION 3.2. *Let f be a continuous self-map of a compact metric space (X, d). If f is iteratively contractive, then f has a unique fixed point $\eta\in X$. Furthermore, if f is nonexpansive, then $\eta=\lim_{n\to\infty} f^n(x)$ for all $x\in X$.*

Proof. For any $x, y\in X$, $fx\neq fy$, there exists an integer $m>0$ such that $d(f^m(x), f^m(y))<d(fx, fy)$. Since $f^m\in C_f$, f has a fixed point η by Theorem A. The uniqueness follows from $d(f^m(x), f^m(y))<d(fx, fy)$.

For the second half, note first that the commutative semigroup $G=\{f^n\}$, $n>0$, is proximally contractive. Hence, by Theorem B, any $x\in X$ and η are proximal, i.e., for any $\mu>0$ there is an integer $N>0$ such that $d(f^N(x), \eta)<\mu$. Since f is nonexpansive, for any $n\geqslant N$, we have $d(f^n(x), \eta)\leqslant d(f^N(x), \eta)<\mu$, i.e., $\eta=\lim_{n\to\infty} f^n(x)$.

The first part of Proposition 3.2 is due to Bailey [1] as a generalization of Edelstein's result [2] for $k=1$.

The following is an easy consequence of Proposition 3.2.

COROLLARY 3.3. *If (X, d) is compact and $f: X\to X$ is contractive, then f has a unique fixed point $\eta\in X$ and for any $x\in X$ we have $\eta=\lim_{n\to\infty} f^n(x)$.*

We need the following.

LEMMA. (Tan [7, Proposition 2.4]) *Let Y be any topological space and $f: Y\to Y$ be a map. If there is an integer $N>0$ and there is a $\zeta\in Y$ such that $\zeta=\lim_{n\to\infty}(f^N)^n(y)$ for each $y\in Y$ then $\zeta=\lim_{n\to\infty} f^n(y)$ for each $y\in Y$.*

Combining Proposition 3.2, Corollary 3.3 and the lemma, we obtain two propositions.

PROPOSITION 3.4. *Let f be a self-map of a compact metric space (X, d). If there is an integer $N>0$ such that f^N is contractive, then f has a unique fixed point $\eta\in X$ and $\eta=\lim_{n\to\infty}f^n(x)$ for each $x\in X$.*

Proposition 3.4 is motivated by Corollary 2.4 of Tan [7]. Note that, in many aspects, contractive maps of compact metric spaces play almost same roles to strict contractions of complete metric spaces.

PROPOSITION 3.5. (Tan [7, Corollary 4.8]) *If (X, d) is compact and $f: X\to X$ is such that for some integer $n>0$, f^n is ~~nonexpansive and iteratively~~ contractive, then f has a unique fixed point $\eta\in X$ and $\eta=\lim_{n\to\infty}f^n(x)$ for all $x\in X$.*

The following is motivated by Corollary 2.6 of [7].

COROLLARY 3.6. *Let f be a self-map of a compact (X, d). If there are maps $R, S: X\to X$ such that $RS=1_X$* and if there exists *an integer $N>0$ such that Sf^NR is either* (1) *contractive or* (2) *nonexpansive and iteratively contractive, then f has a unique fixed point $\eta\in X$ and $\eta=R(\lim_{n\to\infty}(Sf^nR)(x))$ for all $x\in X$.*

Proof. (1) Since $(SfR)^N=Sf^NR$ is contractive, by Proposition 3.4, SfR has a unique fixed point $\zeta\in X$, that is, $(SfR)\zeta=\zeta$. Then $R\zeta=\eta$ is fixed under f. Its uniqueness follows from the surjectivity of R. The last part follows also from Proposition 3.4.

(2) Use Proposition 3.5 instead of Proposition 3.4 in the proof of (1).

4. Common fixed points of families of maps

Let F be a family of self-maps of a set X. An element $x\in X$ is called a *fixed point* of F whenever we have $fx=x$ for every $f\in F$. The following seems to be fundamental.

LEMMA 4.1. *Let F be a family of self-maps of a set X. If a map $g\in F$ commutes with any $f\in F$ and has a unique fixed point, then so does F.*

Proof. Let z be the unique fixed point of g. For any $f\in F$, $gf(z)=fg(z)=gz$, whence $fz=z$ by the uniqueness.

The converse of Lemma 4.1 does not hold, e.g, let $X=[0,1]$ and F be the family of all maps of the form $x\longmapsto ax^n$, $x\in X$, $a>0$, $n=2, 3, \cdots$. Then every $f\in F$ has two fixed points. However, clearly F has a unique fixed point.

Combining Lemma 4.1 and any of propositions or corollaries in §2, we obtain fixed point statements of families of self-maps. Examples of such statements are compact metric space versions of Corollary 2.6 and of Corollary 2.9 in [7].

A rather nontrivial application of Theorem A to a family of maps is the following.

PROPOSITION 4.2. (Holmes [3, Corollary 1 to Theorem 1]) *If* (X, d) *is compact and a commutative semigroup* G *of continuous self-maps of* X *is proximally contractive, then* G *has a unique fixed point.*

Proof. For any $f \in G$ and any $x, y \in X$, $fx \neq fy$, there exists an $h \in G$ such that $d(hx, hy) < d(fx, fy)$. Since $h \in C_f$, by Theorem A, f has a fixed point. We now show, by induction, that any finite subset of G has a common fixed point. Suppose $f_1(z) = f_2(z) = \cdots = f_n(z) = z$ and let f be arbitrary in G. Then, as z and fz are proximal by Theorem B, fore ach $n = 1, 2, 3, \cdots$ there is a $g_n \in G$ such that $d(g_n(z),\ g_n f(z)) < 1/n$. By compactness, there is a subsequence $\{g_{n_i}\}$ of $\{g_n\}$ such that $\{g_{n_i}(z)\}$ converges to a point w, and, so does $\{g_{n_i} f(z)\}$. Therefore $fw = w$ by the continuity and commutativity. But then, for $k = 1, 2, \cdots, n$, $f_k(w) = \lim_{i\to\infty} f_k g_{n_i}(z) = \lim_{i\to\infty} g_{n_i}(z) = w$, and w is a common fixed point of $f_1, \cdots, f_n$. Thus as the set of fixed points of an element of G is closed, compactness insures at least one fixed point of G. Clearly such a point is unique.

The above proof is essentially due to Holmes [3]. Actually, his proof depends on a lemma instead of Theorem A. Note also that the first half of Proposition 3.2 is a consequence of Proposition 4.2.

Let G be a commutative semigroup of self-maps of (X, d). Then G is called *asymptotically contractive* if for any $x, y \in X$, $x \neq y$, there exists $g \in G$ such that $d(gf(x), gf(y)) < d(x, y)$ for any $f \in G$ [3]. Note that if G is asymptotically contractive then it is proximally contractive. Therefore, we have the following.

COROLLARY 4.3. *If* (X, d) *is compact and* G *is asymptotically contractive, then* G *has a unique fixed point.*

Corollary 4.3 is a particular case of Theorem 2 of Holmes [3]. The set $\{z \in X \mid$ there exists $x \in X$ such that for every $f \in G, \varepsilon > 0$, there exists $g \in G$ such that $d(fg(x),\ z) < \varepsilon\}$ is called *the G-closure* of X and is denoted by X^G, and the set $G(z) = \{z\} \cup \{fz \mid f \in G\}$ is called the *orbit* of z [6]. Holmes [3, Theorem 2] showed that if G is asymptotically contractive on a metric space (X, d), then every point of X^G is the unique fixed point of G. Hence, X^G should consist of at most one point. However, this does not guarantee that $X^G \neq \phi$. If X is compact, then for any family F of continuous self-maps of X we have $X^F \neq \phi$ [6, p. 67]. Therefore, we have

COROLLARY 4.4. *In* Corollary 4.3, X^G *consists of exactly one point, which is the unique fixed point of* G.

5. Applications

The last statement has some applications. A commutative semigroup G of continuous self-maps of (X, d) is called *asymptotically nonexpansive* if for every $x, y \in X$, $x \neq y$, there exists $g \in G$ such that $d(gf(x), gf(y)) \leqslant d(x, y)$ for all $f \in G$. Note that if G is asymptotically contractive then G is asymptotically nonexpansive.

(1) Holmes and Narayanaswami [4] showed that if G is asymptotically nonexpansive on a metric space (X, d) and if $z \in X^G$ then for every $f \in G$, $\varepsilon > 0$, there exists $g \in G$ such that $d(fg(z), z) < \varepsilon$.

This is trivial for a compact X and an asymptotically contractive G since X^G consists of the only fixed point of G, by Corollary 4.3.

(2) Holmes (see [6, Theorem 4]) showed that if X is a compact convex subset of a Banach space and G is a commutative semigroup of asymptotically nonxepensive, continuous self-maps of X, then G has a fixed point.
If G is asymptotically contractive then G has a unique fixed point.

(3) Kiang [6, Corollary 2 to Theorem 5] showed that if X is a compact convex subset of a strictly convex Banach space and if G is a commutative semigroup of asymptotically nonexpansive, continuous self-maps of X, then for each $z \in X^G$, the center of $\overline{\mathrm{co}}\, G(z)$ is a fixed point of G.
If G is asymptotically contractive, then X^G consists of the unique fixed point z of G and $X^G = z = \overline{\mathrm{co}}\, G(z) =$ the center of $\overline{\mathrm{co}}\, G(z)$.

References

[1] D. F. Bailey, *Some theorems on contractive mappings*, J. London Math. Soc. **41** (1966), 101-106. MR **32** #6434.

[2] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74-79. MR **24** #A2936.

[3] R. D. Holmes, *On contractive semigroups of mappings.* Pac. J. of Math. **37** (1971), 701-709. MR **46** #8205.

[4] R. D. Holmes and P. P. Narayanaswami, *On asymptotically nonexpansive semigroups of mappings*, Canad. Math. Bull. **13** (1970), 209-214. MR **42**. #1096.

[5] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83** (1976), 261-263.

[6] M. T. Kiang, *Fixed point theorems for certain classes of semigroups of mappings*, Trans. Amer. Math. Soc. **189** (1974), 63-76. MR **50** #14391.

[7] K.-K. Tan, *Fixed point theorems for nonexpansive mappings*, Pac. J. of Math. **41** (1972), 829-842. MR. **47** #2454.

Seoul National University