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NOTE ON PSEUDOUMBILICAL SUBMANIFOLDS OF CODIMENSION 3 WITH NON-ZERO CONSTANT MEAN CURVATURE

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§ 0. Introduction

Let M^n be an n-dimensional submanifold of an m-dimensional Euclidean space E^m $(n<m)$ with the mean curvature vector $H\neq 0$. If the second fundamental tensor in the normal direction H is proportional to the first fundamental tensor of the submanifold M^n, then M^n is said to be pseudoumbilical. The mean curvature vector H is said to be non-parallel if the covariant derivative of H along M^n has non-zero normal component everywhere.

Chen and Yano have recently proved the following:

THEOREM A ([5]). *Let M^n be a pseudoumbilical submanifold of codimension 3 of an $(n+3)$-dimensional Euclidean space E^{n+3} with non-zero constant mean curvature. If the normal connection is trivial and the mean curvature vector is non-parallel, then the submanifold M^n is conformally flat for $n>3$, and consequently M^n is not contained in any hypersphere of E^{n+3} and it is the locus of moving $(n-1)$-spheres where an $(n-1)$-sphere means a hypersphere of a Euclidean n-space.*

The main purpose of the present note is to remove the condition stated in Theorem A that the normal connection is trivial.

§ 1. Preliminaries

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V;\eta^h\}$ and immersed isometrically in E^{n+3} by

$$X: M^n \rightarrow E^{n+3},$$

where X is the position vector from the origin of E^{n+3} to a point of M^n. (In the sequel, the indicies $h, j, i, k, \cdots$ run over the range $\{1, 2, \cdots, n\}$). We identify $X(M^n)$ with M^n itself and represent it by

$$X=X(\eta^1, \eta^2, \cdots, \eta^n).$$

We put $X_i=\partial_i X$, $\partial_i=\partial/\partial\eta^i$, then X_i are n linearly independent vectors

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tangent to M^n and denote by C, D and E three mutually orthogonal unit normals to M^n. Then, denoting by g_{ji} the fundamental metric tensor of M^n, we have

$$g_{ji}=X_j\cdot X_i,$$

where the dot means that the inner product of tangent vectors in E^{n+3}.

Now denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji} we have equations of Gauss for M^n of E^{n+3}

$$\nabla_j X_i=h_{ji}C+k_{ji}D+l_{ji}E, \tag{1.1}$$

where h_{ji}, k_{ji}, l_{ji} are the second fundamental tensors with respect to C, D, E respectively. The mean curvature vector is then given by

$$H=(1/n)g^{ji}\nabla_j X_i, \tag{1.2}$$

where g^{ji} are contravariant components of the metric tensor.

The equations of Weingarten are given by

$$\begin{aligned} \nabla_j C&=-h_j^h X_i+l_j D+m_j E,\\ \nabla_j D&=-k_j^h X_h-l_j C+n_j E,\\ \nabla_j E&=-l_j^h X_h-m_j C-n_j D, \end{aligned} \tag{1.3}$$

where $h_j^h=h_{jt}g^{th}$, $k_{jh}=k_{jt}g^{th}$, $l_j^h=l_{jt}g^{th}$ and l_j, m_j and n_j are third fundamental tensors. We denote the normal components of $\nabla_j C$, $\nabla_j D$ and $\nabla_j E$ by $\nabla_j^\perp C$, $\nabla_j^\perp D$ and $\nabla_j^\perp E$ respectively.

The normal vector field C is said to be parallel if we have $\nabla_j^\perp C=0$, that is, l_j and m_j vanish identically and it said to be non-parallel if $\nabla_j^\perp C$ never vanishes, that is, $l_t l^t+m_t m^t$ never vanishes, where $l^t=l_i g^{it}$, $m^t=m_i g^{it}$.

We have equations of Gauss:

$$K_{kji}{}^h=h_k^h h_{ji}-h_j^h h_{ki}+k_k^h k_{ji}-k_j^h k_{ki}+l_k^h l_{ji}-l_j^h l_{ki}, \tag{1.4}$$

where $K_{kji}{}^h$ is the Riemann-Christoffel curvature tensor, those of Codazzi:

$$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}-m_k l_{ji}+m_j l_{ki}=0, \tag{1.5}$$

$$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}-n_k l_{ji}+n_j l_{ki}=0, \tag{1.6}$$

$$\nabla_k l_{ji}-\nabla_j l_{ki}+m_k h_{ji}-m_j h_{ki}+n_k k_{ji}-n_j k_{ki}=0, \tag{1.7}$$

and those of Ricci:

$$\nabla_k l_j-\nabla_j l_k+h_k^t k_{jt}-h_j^t k_{kt}+m_k n_j-m_j n_k=0, \tag{1.8}$$

$$\nabla_k m_j-\nabla_j m_k+h_k^t l_{jt}-h_j^t l_{kt}+n_k l_j-n_j l_k=0, \tag{1.9}$$

(1.10) $$\nabla_k n_j - \nabla_j n_k + k_k{}^t l_{jt} - k_j{}^t l_{kt} + l_k m_j - l_j m_k = 0.$$

§ 2. Pseudoumbilical submanifolds with non-zero constant mean curuature

Throughout the rest of this paper, we suppose that M^n is a pseudoumbilical submanifold of codimension 3 of an Euclidean space E^{n+3} with non-zero constant mean curvature. Since the mean curvature vector H is nowhere zero, we may choose the first normal C in the direction of H, that is, $H=|H|C$. Then by the assumption we have

(2.1) $$h_{ji} = |H| g_{ji}, \quad |H| = \text{constant} \neq 0,$$

(2.2) $$k_t{}^t = l_t{}^t = 0,$$

where $k_t{}^t = k_{ji} g^{ji}$, $l_t{}^t = l_{ji} g^{ji}$. Thus (1.5) becomes

(2.3) $$l_k k_{ji} - l_j k_{ki} + m_k l_{ji} - m_j l_{ki} = 0.$$

Transvecting (2.3) with g^{ki} and taking account of (2.2), we obtain

(2.4) $$k_{jt} l^t + l_{kt} m^t = 0.$$

Moreover, transvecting (2.3) with l^i and m^i and using (2.4), we find respectively

(2.5) $$l_j{}^t (l_t m_k - m_t l_k) = l_k{}^t (l_t m_j - m_t l_j)$$

and

(2.6) $$k_j{}^t (m_t l_k - l_t m_k) = k_k{}^t (m_t l_j - l_t m_j).$$

If we transvect l^k and m^k to (2.3) and make use of (2.4) (2.6), then we find respectively

(2.7) $$\langle l, l\rangle k_{ji} + \langle m, l\rangle l_{ji} = l_j{}^t (l_t m_i - m_t l_i)$$

and

(2.8) $$\langle m, l\rangle k_{ji} + \langle m, m\rangle l_{ji} = k_j{}^t (m_t l_i - l_t m_i\rangle,$$

where $\langle l, l\rangle = l_t l^t$, $\langle l, m\rangle = l_t m^t$ and $\langle m, m\rangle = m_t m^t$. Eliminating l_{ji} from the last two equalities, we get

(2.9) $$\Delta k_{ji} = (k_{jt} l^t) \{\langle m, m\rangle l_i - \langle m, l\rangle m_i\} + (k_{jt} m^t) \{\langle l, l\rangle m_i - \langle l, m\rangle l_i\},$$

where we have put

(2.10) $$\Delta = \langle l, l\rangle \langle m, m\rangle - \langle m, l\rangle^2.$$

Transvecting (2.7) with $m^j m^i$ and using (2.4), we obtain

(2.11) $$\langle l, l\rangle k(m, m)+\langle m, m\rangle k(l, l)=2\langle l, m\rangle k(l, m).$$

where, $k(l, l)=k_{ji}l^jl^i$, $k(l, m)=k_{ji}l^jm^i$ and $k(m, m)=k_{ji}m^jm^i$.
Transvection (2.9) with l_j and m^j yields respectivly

(2.12) $$\Delta k_{jt}l^t=\{\langle l, m\rangle k(m, l)-\langle l, l\rangle k(m, m)\}l_j + \{\langle l, l\rangle k(m, l)-\langle m, l\rangle k(l, l)\}m_j$$

and

(2.13) $$\Delta k_{jt}m^t=\{\langle m, m\rangle k(m, l)-\langle m, l\rangle k(m, m)\}l_j + \{\langle m, l\rangle k(m, l)-\langle m, m\rangle k(l, l)\}m_j$$

because of (2.11). Thus (2.9) reduces to

(2.14) $$\Delta^2 k_{ji}=\Delta\{-k(m, m)l_jl_i-k(l, l)m_jm_i + k(m, l)(l_jm_i+l_im_j)\}.$$

In the same way, we can verify from (2.4), (2.7) and (2.8) that

(2.15) $$\Delta^2 l_{ji}=\Delta\{l(l, m)(l_jm_i+l_im_j)-l(m, m)l_jl_i - l(l, l)m_jm_i\},$$

where $l(l, l)=l_{ji}l^jl^i$, $l(m, l)=l_{ji}m^jl^i$ and $l(m, m)=l_{ji}m^jm^i$.

Now suppose that $\Delta\neq 0$ on M^n. Then we can suitable choose the normal directions D and E in such a way that there exist mutually orthogonal unit vector fields c^h and e^h such that

(2.16) $$c^h=(l/\sqrt{\langle l, l\rangle})l^h,$$

(2.17) $$e^h=(l/\sqrt{\langle l, l\rangle\Delta})\{\langle l, l\rangle m^h-\langle m, l\rangle l^h\}.$$

From (2.11), (2.16) and (2.17) we have

(2.18) $$k_{ji}c^jc^i+k_{ji}e^je^i=0.$$

If we take account of (2.16) and (2.17), and make use of (2.18) and the fact that $\Delta\neq 0$, then (2.14) reduces to

(2.19) $$k_{ji}=k_2(c_je_i+e_jc_i)+k_1(c_jc_i-e_je_i),$$

where $k_1=k_{ji}c^jc^i$ and $k_2=k_{ji}c^je^i$.

Similarly we can derive from (2.15), (2.16) and (2.17) that

(2.20) $$l_{ji}=l_2(c_je_i+e_jc_i)+l_1(c_jc_i-e_je_i),$$

where $l_1=l_{ji}c^jc^i$ and $l_2=l_{ji}c^je^i$.
From (2.19) and (2.20) we get

(2.21)
$$k_{jt}c^t = k_1 c_j + k_2 e_j, \quad k_{jt}e^t = k_2 c_j - k_1 e_j,$$
$$l_{jt}c^t = l_1 c_j + l_2 e_j, \quad l_{jt}e^t = l_2 c_j - l_1 e_j.$$

Thus, we have from (2.19), (2.20) and (2.21)

(2.22)
$$k_{jt}l_i^t = (k_1 l_1 + k_2 l_2)(c_j c_i + e_j e_i) + (k_1 l_2 - k_2 l_1)(c_j e_i - e_j c_i),$$

from which,

(2.23)
$$k_{ji}l^{ji} = 2(k_1 l_1 + k_2 l_2).$$

Also, from (2.19), (2.20) and (2.21) we have
$$k_{jt}k_i^t = (k_1^2 + k_2^2)(c_j c_i + e_j e_i),$$
$$l_{jt}l_i^t = (l_1^2 + l_2^2)(c_j c_i + e_j e_i),$$

from which,

(2.24)
$$k_j^t k_t^s k_s^i = (k_1^2 + k_2^2) k_j^i,$$
$$l_j^t l_t^s l_s^i = (l_1^2 + l_2^2) l_j^i.$$

It follows from (2.23) and (2.24) that
$$(k_1^2 + k_2^2)(l_1^2 + l_2^2) = (k_1 l_1 + k_2 l_2)^2.$$

Hence we have $k_1 l_2 - k_2 l_1 = 0$. Therefore (2.22) implies that

(2.25)
$$k_{jt}l_i^t - k_{it}l_j^t = 0.$$

Denoting by H_2 and H_3 the symmetric $n\times n$ matrices given by (k_j^h) and (l_j^h) respectively, we see from (2.25) that $H_2 H_3 = H_3 H_2$. Since M^n is pseudoumbilical, the second fundamental tensors $H_1 = (h_j^h)$, $H_2 = (k_j^h)$ and $H_3 = (l_j^h)$ are simultaneousely diagonalizable, that is, the normal connection of M^n in E^{n+3} is trivial.

In the next place we consider the case in which $\Delta = 0$. Moreover we assume that the mean curvature vector H is non-parallel, that is, $\nabla_j^{\perp} C \neq 0$ at a point of M^n. In this case we see that the vector field l^h and m^h are linearly dependent. Thus, there exists a constant A such that $m^h = A l^h$ on M^n. Put $A = \tan\theta$ and
$$\bar{D} = \cos\theta\, D + \sin\theta\, E,$$
$$\bar{E} = -\sin\theta\, D + \cos\theta\, E.$$

Then we see that the third fundamental tensor in the normal direction $\bar{D}$ vanishes. Hence we may assume that $m_j = 0$. Substituting this into (2.3), we obtain

(2.26) $$l_k k_{ji} - l_j k_{ki} = 0,$$

from which, transvecting g^{ji} and using the fact that $k_t{}^t = 0$, we get $k_{jt} l^t = 0$. Thus, we have from (2.26) $l_k k_{ji} k^{ji} = 0$. Since $\nabla_j{}^{\perp} C \neq 0$, we see that $k_{ji} = 0$. Consequently the normal connection is always trivial.

Summing up, we have

PROPOSITION 2.1. *Let M^n be a pseudoumbilical submanifold of codimension 3 of an Euclidean space E^{n+3} with non-zero constant mean curvature. If the mean curvature vector is non-parallel, then the normal connection of M^n in E^{n+3} is trivial.*

From Theorem A and Proposition 2.1 we have

THEOREM 2.2. *Under the same assumptions as those stated in Proposition 2.1, we have the submanifold M^n is conformally flat for $n>3$, and M^n is not contained in any hypersphere of E^{n+3} and it is the locus of moving $(n-1)$-spheres where an $(n-1)$-sphere means a hypersurface of Euclidean n-space.*

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