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ON AN H. PLANAR-GEODESIC CORRESPONDENCE IN A KAEHLERIAN MANIFOLD

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§0. Introduction.

Recently, Y. Tashiro [2] has introduced and investigated H. planar curves and a correspondence along H. planar curves between two φ-connections, where a φ-connection means a symmetricn affie connection with respect to which the Kaehlerian stucture is covariantly constant in a Kaehlerian manifold. He has defined the holomorphically flat correspondence between two φ-connections in a Kaehlerian manifold if it carries H. planar curves to H. planar curves. Moreover he has proved that a Kaehlerian manifold related to a locally Euclidean space under a holomorphically flat correspondence is of constant holomorphically sectional curvature and vice-versa.

In the present paper we consider a φ-connection along a curve in a Kaehlerian manifold and we introduce a correspondence such that an H. planar curve with respect to the φ-connection is always a geodesic with respect to the Levi-Civita connection. The purpose of the present paper is to prove that the Bochner curvature tensor in a Kaehlerian manifold is invariant under above correspondence.

§1. Preliminaries.

Let M be a $2n$-dimensional ($n\geqq 2$) Kaehlerian manifold with a Kaehlerian structure φ_{ji} and metric tensor g_{ji}. A symmetric connection Γ_{ji}^h is called a φ-connection [3] if $\nabla_k\varphi_{ji}=0$, where ∇_k indicates the covariant differentiation with respect to Γ_{ji}^h. We shall denote the curvature tensor, the Ricci tensor and the scalar curvature formed with Γ_{ji}^h by $K_{kji}{}^h$, K_{ji} and K respectively. In a Kaehlerian manifold M, the following identities are satisfied:

$$K_{kji}{}^t\varphi_t{}^h=K_{kjt}{}^h\varphi_i{}^t, \tag{1.1}$$

$$\varphi_j{}^tK_{ti}+\varphi_i{}^tK_{tj}=0. \tag{1.2}$$

In the present paper, we assume that there is a φ-connection Γ_{ji}^h in a Kaehlerian manifold M and we shall denote the curvature tensor and the Ricci tensor of the Levi-Civita connection $'\Gamma_{ji}^h$ formed with g_{ji} by $'K_{kji}{}^h$ and $'K_{ji}$ respectively. We shall also denote the covariant differentiation with respect to $'\Gamma_{ji}^h$ by $'\nabla_k$.

In a Kaehlerian manifold M, a curve $x^i=x^i(t)$ defined by

$$\text{(1.3)}\qquad \frac{d^2x^h}{dt^2}+\Gamma_{ji}{}^h\frac{dx^j}{dt}\frac{dx^i}{dt}-\alpha\frac{dx^h}{dt}=\beta\cdot\varphi_j{}^h\frac{dx^j}{dt}$$

is, by definition, a holomorphically planar curve or an H. planar curve with respect to $\Gamma_{ji}{}^h$, where α and β are certain functions of t.

Denoting the left-member of (1.3) by η^h, we rewrite (1.3) as $\eta^h=\beta\cdot\varphi_j{}^h\xi^j$, where $\xi^j=\frac{dx^j}{dt}$, from which $\beta=-\varphi_{hk}\eta^h\xi^k$, where $\varphi_{hk}=\varphi_h{}^j g_{jk}$. Denoting $\varphi_{ik}\eta^i$ $=2q_k$, we have $\beta=-2q_k\xi^k$. Thus (1.3) is rewritten as

$$\text{(1.3)}'\qquad \frac{d\xi^h}{dt}+\Gamma_{ji}{}^h\xi^j\xi^i=\alpha\cdot\xi^h-(\varphi_j{}^hq_i+\varphi_i{}^hq_j)\xi^j\xi^i.$$

On the other hand a curve $x^i=x^i(t)$ defined by

$$\text{(1.4)}\qquad \frac{d\xi^h}{dt}+{}'\Gamma_{ji}{}^h\xi^j\xi^i=\gamma\xi^h$$

is, by definition, a geodesic with respect to ${}'\Gamma_{ji}{}^h$ in the Kachlesian manifold M, where γ is a certain function of t.

In a Kaehlerian manifold admitting two φ-connections, if any H. planar curve with respect to one of the connection is always a geodesic with respect to the other connection, we say this correspondence between given two connections an *H. planar-geodesic correspondence.*

By standard arguments, it follows from (1.3)′ and (1.4) that an H. planar-geodesic correspoodence is expressed by the relation:

$$\text{(1.5)}\qquad {}'\Gamma_{ji}{}^h=\Gamma_{ji}{}^h+(p_j+c\tilde{q}_j)\delta_i^h+(p_i+c\tilde{q}_i)\delta_j^h+q_j\varphi_i{}^h+q_i\varphi_j{}^h,$$

where p_i is a gradient vector, $\tilde{q}_j=\varphi_j{}^tq_t$, q_t being the covector defined in (1.3)′ and $c=-1/(2n+1)$.

Then the curvature tensors are related to each other by

$$\text{(1.6)}\qquad \begin{aligned}{}'K_{kji}{}^h=K_{kji}{}^h&-\delta_k^hP_{ji}+\delta_j^hP_{ki}+\delta_i^h(P_{kj}-P_{jk})\\&-\varphi_k{}^hQ_{ji}+\varphi_j{}^hQ_{ki}+\varphi_i{}^h(Q_{kj}-Q_{jk}),\end{aligned}$$

where we have put

$$\text{(1.7)}\qquad \begin{aligned}P_{ji}=\nabla_jp_i-p_jp_i-(q_j\tilde{p}_i+q_i\tilde{p}_j)+c\nabla_j\tilde{q}_i-c(p_j\tilde{q}_i+p_i\tilde{q}_j)\\+(2c-1)q_jq_i-c^2\tilde{q}_j\tilde{q}_i,\end{aligned}$$

where $\tilde{p}_j=\varphi_j{}^tp_t$ and

$$\text{(1.8)}\qquad Q_{ji}=\nabla_jq_i-q_j\tilde{q}_i-q_i\tilde{q}_j.$$

§2. An H. planar-geodesic correspondence in a Kaehlerian manifold.

In this sectinn, we consider an H. planar-geodesic correspondence defined in the previous section.

Substituting (1. 5) into $'\nabla_k\varphi_{ji}=0$ and taking account of the fact that the connection Γ_{ji}^h also a φ-connection, we have

$$[\tilde{p}_j-(1-c)q_j]\delta_k^i-[p_j-(1-c)\tilde{q}_j]\varphi_k^i=0. \tag{2.1}$$

Contracting with respect to i and k in (2. 1), we obain

$$\tilde{p}_j=(c-1)q_j, \qquad p_j=(1-c)\tilde{q}_j, \tag{2.2}$$

from which

$$\nabla_j\tilde{q}_i-\nabla_i\tilde{q}_j=0. \tag{2.3}$$

Taking account of (2. 2), (2. 3) and (1. 7), we obtain

$$P_{ji}-P_{ij}=0. \tag{2.4}$$

Substituting (2. 2) into (1. 7), we obtain

$$P_{ji}=\nabla_j\tilde{q}_i+q_jq_i-\tilde{q}_j\tilde{q}_i. \tag{2.5}$$

Taking account of (2. 5), (1. 7) and (1. 8), we easily obtain

$$P_{ji}=\varphi_i^tQ_{jt}, \qquad Q_{ji}=-\varphi_i^tP_{jt}. \tag{2.6}$$

On the other hand, contracting with respect to h and k in (1. 6) and taking account of (2. 4) and (2. 6), we have

$$'K_{ji}=K_{ji}-2nP_{ji}+\varphi_j^tQ_{ti}+\varphi_i^tQ_{tj}. \tag{2.7}$$

Since both of the connections Γ_{ji}^h and $'\Gamma_{ji}^h$ are φ-connections, both of $'K_{ji}$ and K_{ji} satisfy the relation (1. 2).

Substituting (2. 7) into (1. 2) and taking account of the fact that stated above and (2. 6), we obtain

$$Q_{ji}+Q_{ij}=0, \tag{2.8}$$

proveded that $n\geqq 2$.

Taking account of (2. 8) and (2. 6), (2. 7) is rewritten as

$$'K_{ji}=K_{ji}-2(n+1)P_{ji}. \tag{2.9}$$

§3. Properties of an H. planar-geodesic correspondence in a Kaehlerian manifold.

A $2n$-dimensional Riemannian manifold $'M$ with the metric tensor g_{ji}, the curvature tensor $'K_{kji}^h$, the Ricci tensor $'K_{ji}$ and the scalar curvature $'K$

satisfying the relation:

$$'K_{kji}{}^h+\frac{1}{2n-2}('K_{ki}\delta_j^h-'K_{ji}\delta_k^h+g_{ki}'K_j{}^h-g_{ji}'K_k{}^h) \qquad (3.0)$$

$$-\frac{'K}{(2n-1)(2n-2)}(g_{ki}\delta_j^h-g_{ji}\delta_k^h)=0,$$

where $'K_j{}^h='K_{ji}g^{ih}$, is called conformally flat.

Let a Kaehlerian manifold M with the curvature tensor $K_{kji}{}^h$, the Ricci tensor K_{ji} and the scalar curvature K be related locally to a conformally flat space under an H. planar-geodesic correspondence. In this case, substituting (1. 6) and (2. 9) into (3. 0), taking account of (2. 4), (2. 6) and (2. 8), and contracting with respect to h and k in it, we easily obtain $P_{ji}=0$, from which $Q_{ji}=0$ with the help of (2. 6). Therefore we find that

$$'K_{kji}{}^h=K_{kji}{}^h, \qquad 'K_{ji}=K_{ji}, \qquad 'K=K.$$

Thus we have the following

THEOREM 1. *A Kaehlerian manifold of dimension* $\geqq 4$ *related locally to a conformally flat space under an H. planar-geodesic correspondence is also conformally flat.*

Next, we investigate that whethere or no a Kaehlerian manifold can be related lacally to a projectively flat space under an H. planar-geodesic correspondence.

If a $2n$-dimensional Riemannian manifold $'M$ with the curvature tensor $'K_{kji}{}^h$ and the Ricci tensor $'K_{ji}$ satisfy the relation:

$$'K_{kji}{}^h+\frac{1}{2n-1}('K_{ki}\delta_j^h-'K_{ji}\delta_k^h)=0, \qquad (3.1)$$

then $'M$ is called projectively flat.

Let a Kaehlerian manifold M with the curvature tensor $K_{kji}{}^h$ and the Ricci tensor K_{ji} be related locally to a projectively flat space under an H. planar-geodesic correspondence. In this case, substituting (1. 6) and (2. 9) into (3. 1) and taking account of (2. 4) and (2. 8), we obtain

$$K_{kji}{}^h-\frac{1}{2n-1}\delta_k^h(K_{ji}-3P_{ji})+\frac{1}{2n-1}\delta_j^h(K_{ki}-3P_{ki}) \qquad (3.2)$$

$$-\varphi_k{}^hQ_{ji}+\varphi_j{}^hQ_{ki}+2\varphi_i{}^hQ_{kj}=0.$$

Substituting (3. 2) into (1. 1) and taking account of (2. 6), we obtain

$$\frac{1}{2n-1}[\varphi_k{}^h(K_{ji}-3P_{ji})-\varphi_j{}^h(K_{ki}-3P_{ki})]-\delta_k^hQ_{ji}+\delta_j^hQ_{ki} \qquad (3.3)$$

$$=\frac{1}{2n-1}[\delta_k^h(K_{jt}-3P_{jt})\varphi_i{}^t-\delta_j^h(K_{kt}-3P_{kt})\varphi_i{}^t]+\varphi_k{}^hP_{jt}-\varphi_j{}^hP_{ki}.$$

Contracting with respect to h and k in (3.3) and taking account of (1.2) and (2.6), we find

$$\varphi_j{}^t K_{ti} = 2(n+1) Q_{ji}. \tag{3.4}$$

Substituting (3.4) into the first equation of (2.6), we find

$$K_{ji} = 2(n+1) P_{ji}. \tag{3.5}$$

Substituting (3.5) into (2.9), we find $'K_{ji}=0$, from which $'K_{kji}{}^h=0$ because of (3.1). Thus we have the following

THEOREM 2. *A Kaehlerian manifold of dimension $\geqq 4$ cannot be related locally to a projectively flat space under an H. planar-geodesic correspondence.*

It is well known that an n (>1)-dimensional Riemannian manifold $'M$ is projectively flat if $'M$ is of constant sectional curvature and vice-versa.

Taking account of above fact we have the following

COROLLARY. *A Kaehlerian manifold of dimension $\geqq 4$ cannot be related locally to a space of constant sectional curvature $'k(\neq 0)$ under an H. planar-geodesic correspondence.*

Substituting (2.4) and (2.8) into (1.6), we see that the curvature tensors are related to each other by

$$'K_{kjih} = K_{kjih} - g_{kh}P_{ji} + g_{jh}P_{ki} - \varphi_{kh}Q_{ji} + \varphi_{jh}Q_{ki} + 2\varphi_{ih}Q_{kj}, \tag{3.6}$$

under an H. planar-geodesic correspondence.

From (3.6) and the well known fact:

$$'K_{kjih} = {}'K_{ihkj}, \qquad K_{kjih} = K_{ihkj},$$

we find with the help of (2.6)

$$P_{ji} = \frac{\alpha}{2n} g_{ji}, \qquad Q_{ji} = \frac{\alpha}{2n} \varphi_{ji}, \tag{3.7}$$

where we have put $\alpha = P_t{}^t = P_{ti}g^{it}$.

Substituting (3.7) into (3.6), we have

$$'K_{kjih} = K_{kjih} - \frac{\alpha}{2n}(g_{kh}g_{ji} - g_{jh}g_{ki} + \varphi_{kh}\varphi_{ji} - \varphi_{jh}\varphi_{ki} - 2\varphi_{kj}\varphi_{ih}). \tag{3.8}$$

Thus we have the following [2]

THEOREM 3. *A Kaehlerian manifold of dimension $\geqq 4$ related locally to a Euclidean space under an H. planar-geodesic correspondence is locally Fubinian.*

§ 4. A characterization of the Bochner curvature tensor.

We now consider the so-called Bochner curvature tensor [4] defind by

$$(4.1)\qquad B_{kjih}=K_{kjih}+g_{kh}L_{ji}-g_{jh}L_{ki}+g_{ji}L_{kh}-g_{ki}L_{jh}+\varphi_{kh}M_{ji}-\varphi_{jh}M_{ki}+\varphi_{ji}M_{kh}-\varphi_{ki}M_{jh}-2(\varphi_{ih}M_{kj}+\varphi_{kj}M_{ih}),$$

where

$$(4.2)\qquad L_{ji}=-\frac{1}{2n+4}K_{ji}+\frac{1}{2(2n+2)(2n+4)}Kg_{ji},$$

$$(4.3)\qquad M_{ji}=-L_{jt}\varphi_i^{\ t}.$$

Substituting (3.7) into (2.8), the Ricci tensor and the scalar curvature are related to each other by

$$(4.4)\qquad 'K_{ji}=K_{ji}-\frac{1}{n}(n+1)\alpha g_{ji},$$

$$(4.5)\qquad 'K=K-2(n+1)\alpha,$$

under an H. planar-geodesic correspondence.

Taking account of (4.4) and (4.5), we find that the tensors defind by (4.2) and (4.3) are related to each other by

$$(4.6)\qquad 'L_{ji}=L_{ji}+\frac{\alpha}{4n}g_{ji},$$

$$(4.7)\qquad 'M_{ji}=M_{ji}+\frac{\alpha}{4n}\varphi_{ji},$$

under an H. planar-geodesic correspondence.

Substituting (3.8), (4.6) and (4.7) into (4.1), we find that

$$'B_{kjih}=B_{kjih},$$

where $'B_{kjih}$ is the Bochner curvature tensor formed with the connection $'\Gamma_{ji}^{h}$.

Thus we have the following

THEOREM 4. *The Bochner curvature tensor is invariant under an H. planar-geodesic correspondence in a Kaehlerian manifold.*

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