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ON PRO-AFFINE ALGEBRAIC GROUPS(*)
(An approach from Hopf algebras, and applications)

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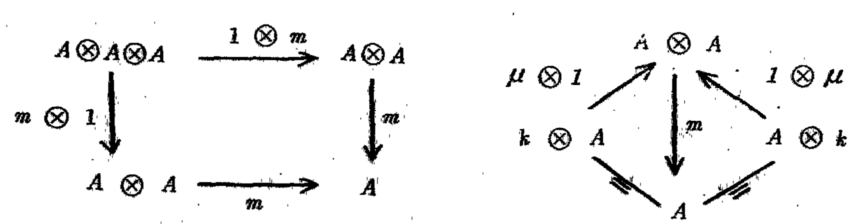
§ 1. Introduction.

The theory of affine algebraic groups, initiated by Chevalley ([3]) in connection of Lie group theory, where these groups appeared as algebraic linear groups, has been further developed by Borel, Steinberg, Tits and others over the last two decades and has made significant contributions in such areas as representation theory of Lie groups, finite simple groups, classical linear groups, etc., Mostow and Hochschild on the other hand have effectively fused techniques in the representation theory with the theory of affine algebraic groups so as to provide an efficient tool especially for the study of Lie groups, and this naturally has led us to consider pro-affine algebraic groups, certain projective limits of affine algebraic groups. The main purpose of this paper is to introduce the notion of pro-affine algebraic groups developed by Hochschild and Mostow in ([7]), which is based on Hopf algebras of representative functions, and then to cite some of its applications.

Throughout this paper, k will always denote a field, and all tensor products are taken over k.

§ 2. Hopf algebras.

We recall that an *algebra over* k (or simply a k-algebra) may be defined as a triple (A, m, μ) with A a k-linear space, $m: A\otimes A\to A$ a k-linear map called *multiplication*, $\mu: k\to A$ a k-linear map called the *unit* (*map*), such that the diagrams



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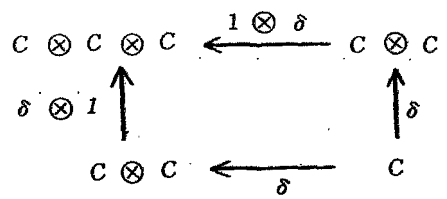
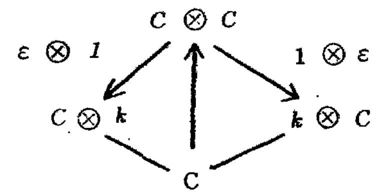
are commutative.

A formal dualization of the diagram above (i. e., turning all the arrows around) immediately leads to the definition of coalgebras:

DEFINITION. A *coalgebra* over k (or simply k-*coalgebra*) is a triple (C, δ, ε) with C a k-linear space, $\delta : C \to C \otimes C$ a k-linear map called *comultiplication* and $\varepsilon : C \to k$ a k-linear map called the *counit* (*map*), which satisfies the diagram dual to those of a k-algebra, namely

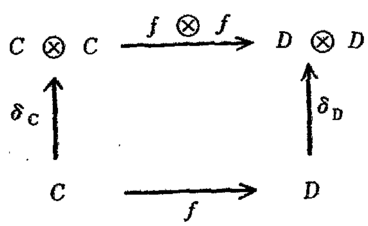


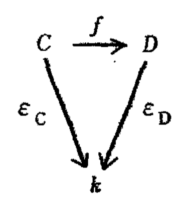
(*coassociativity*)



(*counital map*)

A *morphism* of a k-coalgebra C to a coalgebra D is a k-linear map $f : C \to D$ such that the diagrams





are commutative.

A subspace V of a k-coalgebra $C = (C, \delta, \varepsilon)$ is called a *sub-coalgebra* of C if $\delta(V) < V \otimes V$, and, in this case, the triple $(V, \delta | V, \varepsilon | V)$ is the coalgebra and the inclusion $V \to C$ is a morphism of coalgebras.

DEFINITION. Let (C, δ, ε) be a k-coalgebra and let (A, m, μ) be a k-algebra. For k-linear maps $f, g : C \to A$, define

$$f*g : C \to A$$

by $f*g = m\circ(f\otimes g)\circ\delta : C \xrightarrow{\delta} C\otimes C \xrightarrow{f\otimes g} A\otimes A \xrightarrow{m} A$ Then $f*g$ is a k-linear map, called the *convolution* of f and g.

The k-linear space $\mathrm{Hom}_k(C, A)$ of all k-linear maps $C \to A$ becomes a k-algebra with algebra product being the convolution, in which the unit map is $\mu\circ\varepsilon : C \to k \to A$.

DEFINITION. A *Hopf algebra* over k is a k-algebra E together with k-

algebra morphisms

$$\delta : E \to E \otimes E, \quad \varepsilon : E \to k$$

such that (E, δ, ε) is a coalgebra.

A *morphism* of *Hopf algebras* is a k-algebra morphism $\rho : E_1 \to E_2$ which is also a morphism of coalgebras. A subspace V of a Hopf algebra E is called a *Hopf subalgebra* if it is a subalgebra of E (as a k-algebra) and a sub-coalgebra (as a k-coalgebra).

DEFINTION. Given a Hopf algebra over k, the two-sided inverse η to the identity map I in the convolution algebra $\mathrm{Hom}_k(E, E)$ is called the *antipode* for E. Thus antipode is a k-linear map $\eta : E \to E$ such that $I * \eta = \mu \circ \varepsilon = \eta * I$, μ and ε being the unit and the counit of E.

Let E be a Hopf algebra. Regarding k as a k-algebra in which the multiplication $k \otimes k \to k$ is treated as an identification, we obtain the convolution algebra $\mathrm{Hom}_k(E, k)$. The convolution $x * y$, for $x, y \in \mathrm{Hom}_k(E, k)$, is given by $x * y = (x \otimes y) \circ \delta$, where δ is the comultiplication of E, and the unit of this algebra is the counit $\varepsilon : E \to k$ of the Hopf algebra E.

We now assume that E has the antipode η. Then a k-algebra morphism $x : E \to k$ is invertible in the convolution algebra $\mathrm{Hom}_k(E, k)$ with its inverse being $x \circ \eta$, and the set $\mathrm{Hom}_{k\text{-}alg}(E, k)$ of all k-algebra morphisms $E \to k$ is a ubgroup of the group of all invertible elements of the convolution salgebra $\mathrm{Hom}_k(E, k)$. We denote this group by $\mathcal{G}(E)$.

In order to describe the group $\mathcal{G}(E)$ as a subgroup of the general linear group $GL_k(E)$, we define the following notion of proper maps.

DEFINITION. A k-linear map $\sigma : E \to E$ is called proper if the diagram

$$\begin{array}{ccc} E \otimes E & \xrightarrow{1 \otimes \sigma} & E \otimes E \\ \delta \Big| & & \Big| \delta \\ E & \xrightarrow[\sigma]{} & E \end{array}$$

is commutative.

Then the map $\sigma \to \varepsilon \circ \sigma$ (ε=counit of E) defines an isomorphism of the group of all proper k-algebra automorphisms $\mathrm{Aut}_p(E)$ of E onto the group $\mathcal{G}(E)$.

§ 3. Hopf algebra of representative functions.

DEFINITION. Let G be an arbitrary group, and for a function $g : G \to k$ and

$x \in G$, define the *left* (resp. *right*) *translate* $x \cdot g$ (resp. $g \cdot x$) of g by $(x \cdot g)(y) = g(yx)$ (resp. $(g \cdot x)(y) = g(xy)$) for all $y \in G$.
A function $g : G \to k$ is called *representative* if all its translates lie in a finite dimensional space of functions.

The representative functions on G form a k-algebra $R(G)$ with usual product. The group multiplication $G \times G \to G$ defines a map $\gamma : R(G) \to R(G \times G)$ given by

$$\gamma(f)(x, y) = f(xy)$$

and we see that $\gamma(f)$ is actually in the subalgebra $R(G) \otimes R(G)$ of $R(G \times G)$, where $R(G) \otimes R(G)$ is regarded as a subalgebra of $R(G \times G)$ so that $(g \otimes h)(x, y) = g(x)h(y)$ for $g, h \in R(G)$.

Hence we obtain a Hopf algebra structure on $R(G)$ with γ being the comultiplication and the counit being given by the evaluation $\varepsilon : R(G) \to k$ at 1. This Hopf algebra has an antipode, namely, the map $\eta : R(G) \to R(G)$ defined by $\eta(f)(x) = f(x^{-1})$, $x \in G$ and $f \in R(G)$.

DEFINITION. A subalgebra A of the k-algebra $R(G)$ is called *fully stable* if it is stable under the left and right translations by elements of G and also stable under the antipodal map $\eta : R(G) \to R(G)$. Note that a fully stable subalgebra A of $R(G)$ is a Hopf subalgebra of $R(G)$ which has antipode, and a k-linear map $\sigma : A \to A$ is proper if and only if $\sigma(g \cdot x) = \sigma(g) \cdot x$ for all $x \in G$ and $g \in A$.

§4. Pro-affine algebraic groups.

DEFINITION. A *pro-affine algebraic group* over k is a group G together with a Hopf subalgebra A of $R(G)$ which is invariant under the antipodal map η such that the canonical map

$$G \to \mathcal{G}(A)$$

sending each $x \in G$ to $x^\circ \in \mathcal{G}(A)$, where $x^\circ : A \to k$ is the evaluation map given by $x^\circ(f) = f(x)$ for $f \in A$, is a bijection.

A is called the *Hopf algebra of polynomial functions* on G and is deoted by $\mathcal{A}(G)$. A *morphism of pro-affine algebraic groups* is a group homomorphism $\rho : G \to H$ such that if $f \in \mathcal{A}(H)$, then $f \circ \rho \in \mathcal{A}(G)$. Note in this case that the map $f \to f \circ \rho : \mathcal{A}(H) \to \mathcal{A}(G)$ is a morphism of Hopf algebras.

REMARKS. (1) If $\mathcal{A}(G)$ is finitely generated as a k-algebra, then G is a usual affine algebraic group over k ([2], [5]).

(2) A pro-affine algebraic group G over an algebraically closed field k is a projective limit of affiine algebraic groups. (This follows from the fact that

$\mathcal{A}(G)$ is a reduced (i. e., 0 is the only nilpotent element of $\mathcal{A}(G)$) commutative Hopf algebra, and as such it is the union of the family of finitely generated fully stable subalgebras of $\mathcal{A}(G)$ ([13]). Moreover, if A is any finitely generated fully stable subalgebra of $\mathcal{A}(G)$, then $\mathcal{G}(A)$ is an affine algebraic group ([5]), and for every pair A_1, A_2 of finitely generated fully stable subalgebras of $\mathcal{A}(G)$ with $A_1 < A_2$, we have the corresponding affine algebraic groups $\mathcal{G}(A_1)$ and $\mathcal{G}(A_2)$, and the restriction map $\mathcal{G}(A_2) \to \mathcal{G}(A_1)$, which is surjective (because k is algebraically closed). Thus $\mathcal{G}(\mathcal{A}(G)) = G$ is the projective limit of the system of the groups $\mathcal{G}(A)$ and closed maps $\mathcal{G}(A_2) \to \mathcal{G}(A_1)$.)

(3) The argument similar to the proof of (2) leads to the following result ([7]) which is known in the affine case ([5]).

THEOREM. *Let A be a reduced commutative Hopf algebra with autipode and assume that the ground field k is algebraically closed. Then the group $\mathcal{G}(A)$ is a pro-affine algebraic group with $\mathcal{A}(\mathcal{G}(A)) = A$.*

DEFINITION. For a subgroup H of a pro-affine algebraic group G, let $\mathcal{I}(H)$ denote the annihilator of H in $\mathcal{A}(G)$ (i. e., the ideal of $\mathcal{A}(G)$ consisting of f such that $f|H=0$), and let H^* be the annihilator of $\mathcal{I}(H)$ in G, i. e., the subgroup of G consisting of the elements $x \in G$ with $f(x) = 0$ for all $f \in \mathcal{I}(H)$. Then H^* inherits the structure of apro-affine algebraic group, with $\mathcal{A}(H^*)$ being identified with $\mathcal{A}(G)/\mathcal{I}(H)$. H^* is called the *algebraic hull* of H in G, and we say that H is an *algebraic subgroup* of G if $H = H^*$.

Given a normal algebraic subgroup H of pro-affine algebraic group G, the quotient group G/H inherits the structure of a pro-affine algebraic group, with $\mathcal{A}(G/H)$ being identified with $\mathcal{A}(G)^H$, the H-fixed (under left translations) part of $\mathcal{A}(G)$. The canonical map $G \to G/H$ is then a morphism and satisfies the usual universal property of the quotient group.

Most of the results in affine algebraic groups may be extended to the pro-affine case. Below we discuss some of such results concerning the structure of pro-affine algebraic groups.

(i) First we shall consider the decomposition into components of a pro-affine algebraic groups. We say that a pro-affine algebraic group G is *connected* if $\mathcal{A}(G)$ is an integral domain and is *pro-finite* if $\mathcal{A}(G)$ is the union of the family of finite-dimensional fully stable subalgebras, so that G appears as the projective limit of finite groups.

THEOREM ([5]). *Let G be a pro-affine algebraic group over an algebraically closed field k. There is a normal connected algebraic subgroup G_1 of G*

such that G/G_1 is pro-finite.

(ii) Next we shall consider the Mostow decomposition into the unipotent radical and a maximal reductive subgroup.

A subgroup K of an affine algebraic group G is called *unipotent* if, for every finite-dimensional left stable subspace V of $\mathcal{A}(G)$, the representation of K by left translations on V is unipotent. There is a normal unipotent subgroup G_u of G that contains every normal unipotent subgroup of G. G_u is called the *unipotent radical* of G. G_u is an algebraic subgroup of G. A subgroup of G is called *reductive* if its representatioin by left translations on $\mathcal{A}(G)$ is semisimple. Then we have:

THEOREM ([7]). *Let G be an affine algebraic group over an algebraically closed field k of characteristic 0. There is a reactive algebraic subgroup K of G such that G is the semidirect product $G_u \cdot K$. Moreover, if L is any reductive subgroup of G, then there is an element t in $[G, G_u]^*$ such that $t\,Lt^{-1} < K$.*

(iii) The notion and the structure of solvable affine algebraic groups may be extended to the pro-affine case. A subgroup H of a pro-affine algebraic group G is called *pro-solvable* (resp. *pro-toroid*) if, for every finitely generated fully stable subalgebra T of $\mathcal{A}(G)$, the image of $\mathcal{G}(\mathcal{A}(G))=G$ in $\mathcal{G}(T)$ under the restriction map $\mathcal{G}(\mathcal{A}(\mathcal{G}))\to\mathcal{G}(T)$ under the restriction map $\mathcal{G}(\mathcal{A}(G))\to\mathcal{G}(T)$ is solvable (resp. a toroid).

THEOREM ([10]). *Let G be a connected pro-affine algebraic group over an algebraically closed field k. Then there exists a pro-toroid K such that G is the semidirect product $G_u \cdot K$. Moreover, if L is any, there exists $t \in G_u$ such that $tLt^{-1} < K$.*

For results concerning Borel subgroups and others, see [10].

§ 5. Universal algebraic hull.

Let $\mathcal{C}$ be a category of groups such that the full linear group $GL(V)$ of every finite-dimensional vector space V over k is in $\mathcal{C}$.

For every representation $\rho: G\to GL(V)$, dim $V<\infty$, in $\mathcal{C}$, let $[\rho]$ denote the k-linear span of the coefficient functions (i. e., functions of the form $\lambda\circ\rho$, where λ is a k-linear functional on $\mathrm{End}_k(V)$), and we let $\mathcal{C}(G)=U_\rho[\rho]$, where ρ runs over all finite-dimensional representations of G in $\mathcal{C}$.

$\mathcal{C}(G)$ is a fully stable subalgebra of $R(G)$, and hence is a Hopf algebra with the antipodal map. We may also view $\mathcal{C}(G)$ as a universal representation space of G in the sense that if $\rho: G\to GL(V)$, dim $V<\infty$, is a representation in $\mathcal{C}$, then there exists an injective k-linear map $V\to\mathcal{C}(G)$ which

is equivariant under the action of G. The pro-affine algebraic group $G^*=\mathcal{G}(\mathcal{C}(G))$ is called the ***universal algebraic hull*** of G. (The term "universal" may be justified as follows:
We have seen in §2 that $G^*=\mathcal{G}(\mathcal{C}(G))$ is canonically isomorphic with $\mathrm{Aut}_p(\mathcal{C}(G))$, the group of all proper automorphisms of $\mathcal{C}(G)$. Identifying G^* with $\mathrm{Aut}_p(\mathcal{C}(G)$, G^* then consists of all k-algebra automorphisms $\sigma : \mathcal{C}(G) \to \mathcal{C}(G)$ such that $\sigma : \mathcal{C}(G) \to \mathcal{C}(G)$ such that $\sigma(f \cdot x) = \sigma(f) \cdot x$ for $f \in \mathcal{C}(G)$ and $x \in G$. For every finite-dimensional representation $\rho : G \to GL(V)$ in $\mathcal{C}$, view V as a finite dimensional subspace of $\mathcal{C}(G)$ stable under left and right translations. Then the restriction $G_V{}^*$ of G^* to V is isomorphic with the usual Zariski closure $\rho(G)^*$ of $\rho(G)$ in $GL(V)$.) we have, $G^*=\varprojlim G_V{}^*$, where V runs over the directed set of finite-dimensional stable subspaces of $\mathcal{C}(G)$.

The unipotent radical $U(G)$ of G^* is called the ***unipotent hull*** of G.

The construction of G^* above can be carried out, for example, in any of the following categories $\mathcal{C}$.

1. Discrete groups, all finite-dimensional representations.
2. Lie groups with finitely many connected components, all finite-dimensional analytic representations.
3. Topologcial groups, all finite-dimensional continuous representations.

§ 6. Applications.

Below we cite some of the applications of the pro-affine algebric group theory (or, to be more precise, the applications of the functor $G \to G^*$).

(1) Tanaka Duality ([4]).

If G is a compact group, Tanaka duality in Harmonic Analysis states in our language that the canonical map $\tau : G \to G^*$ is an isomorphism (of topological groups), where $G^*=\mathrm{Hom}_{k\text{-}alg}(\mathcal{C}(G), R)$ is equipped with the finite-open topology. For Tanaka Duality in Lie groups, see ([8]).

(2) Discrete Subgroups of Lie Groups.

Malcev proved in [9] that a finitely generated torsion-free nilpotent group D can be embedded into a simply connected nilpotent real analytic group G as a uniform subgroup (i. e., G/D compact), and that D determines G uniquely.

Although a discrete uniform subgroup D of a simply connected solvable real analytic group G does not always determine G, their unipotent hulls $U(D)$ and $U(G)$ are naturally isomorphic, and G can be described in terms of $U(G)$.

If G is a polycyclic group (resp. simply connected solvable real analytic group), then the unipotent hull $U(G)$ is finite-dimensional, and, in fact,

$\dim_C U(G) = \mathrm{rank} G$ (resp. dim G) ([12]). (Our Malcev embedding mentioned above is merely the canonical map $D \to U(D)$.)

Using this result and exploiting the isomorphism $U(D)$ $U(G)$, Mostow ([12]) reproved the conjecture of P. Hall stating that every polycyclic group has a faithful representation by integral matrices. (This conjecture was first proved by L. Auslander in [1]).

(3) COMPLEXIFICATION OF REAL ANALYTIC GROUPS.

A *universal complexification* of a real analytic group G is a coutiuuous homomorphism τ of G iuto a complex aualytic group G_C which is *universal* iu the sense that if η is a continuous homomorphism of G into a complex analytic group H, there is a unique morphism $\eta' : G_C \to H$ of complex analytic groups such that $\eta' \circ \tau = \eta$.

The notion of the universal complexification plays a central role in the representation theory of Lie groups ([4]), and its construction becomes more transparent under the G^* setting. Let $\tau : G \to G^*$ be the canonical map. Then τ induces a map τ of the Lie algebra $\mathcal{L}(G)$ of G into derivations of $\mathcal{A}(G)$, and thus $L_C = \tau(\mathcal{L}(G)) \otimes_R C$ is a complex Lie algebra of derivations of $\mathcal{A}(G)$.

Corresponding to L_C, there is a complex analytic subgroup G_C of G^*. Then $\tau(G) < G_C$ and $\tau : G \to G_C$ is a universal complexification.

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