

CURVATURE TENSORS

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0. Introduction.

Singer and Thorpe [5] established a natural decomposition of curvature tensors [1] on an n -dimensional real vector space with inner product and Nomizu [4] used this decomposition to study generalized curvature tensor fields; in particular he studied proper tensor fields on a Riemannian manifold. To make the theory of submanifolds in conformal differential geometry more up to date, Kowalski [2] considered also a decomposition theory. In these papers the Weyl conformal curvature tensor is obtained in a very natural way as a projection of the Riemann curvature tensor.

In order to study the well known Bochner curvature tensor (a complex analogue of the Weyl tensor) Sitaramayya [6] and Mori [3] gave a similar decomposition for K -curvature tensor fields on a Kähler manifold.

In [7], [8] we showed that the same method can be used to define a Bochner curvature tensor on a large class of almost Hermitian manifolds which contains the Kähler manifolds.

But some other curvature tensors play also a more or less important role in differential geometry. For example, the concircular and projective curvature tensors are well known and their complex analogues have already been considered. The main purpose of this paper is to show how the same method applies to derive these tensors as (orthogonal) projections of a given curvature tensor. It is also plausible that this method may be used to extend the results to other interesting classes of differentiable manifolds.

1. The concircular curvature tensor.

Let V be an n -dimensional ($n > 1$) real vector space with inner product g . A tensor L of type $(1, 3)$ over V is a bilinear mapping $L : V \times V \rightarrow \text{Hom}(V, V) : (x, y) \mapsto L(x, y)$. With this tensor we associate another tensor L_R of type $(0, 2)$ which is the bilinear function defined by

$$L_R(x, y) = \text{trace}(z \in V \mapsto L(z, x)y \in V).$$

The corresponding tensor $Q=Q(L)$ of type $(1, 1)$ is given by $L_R(x, y)=g(Qx, y)$ and L_R , as well as Q , are called the *Ricci tensor* associated with L . The trace of Q is called the *scalar curvature* $l=l(L)$ of L .

We say that L is a *quasi-curvature tensor* if $L(x, y)=-L(y, x)$.

Denote by $\mathcal{L}(V)$ the vector space of all quasi-curvature tensors over V . This is a subspace of the tensor space of type $(1, 3)$ over V and has a natural inner product induced from that on V :

$$\langle L, \tilde{L} \rangle = \text{trace } \tilde{L}^t \circ L = \sum_{i, j, k=1}^n g(L(e_i, e_j)e_k, L(e_i, e_j)e_k),$$

$\{e_i\}$ being an orthonormal basis.

Now we define two maps which are clearly *linear*:

$$h : \mathcal{R} \rightarrow \mathcal{L}(V) : \alpha \mapsto \frac{1}{2}\alpha L_{I, I},$$

$$\mathfrak{C} : \mathcal{L}(V) \rightarrow \mathcal{R} : L \mapsto l.$$

\mathfrak{C} is called the *scalar contraction map* and $L_{I, I}(x, y)=2x \wedge y$ is the skew-symmetric endomorphism of V defined by $(x \wedge y)z=g(y, z)x-g(x, z)y$. Clearly $x \wedge y$ is a quasi-curvature tensor.

LEMMA 1. $\mathfrak{C} \circ h$ is an isomorphism.

Proof. It is sufficient to show that $\mathfrak{C} \circ h$ is injective. Let $\alpha \in \mathcal{R}$ with $\mathfrak{C}h\alpha=0$. This implies $\alpha n(n-1)=0$ and since $n \geq 2$, we obtain $\alpha=0$.

LEMMA 2. $\text{Im } h$ is orthogonal to $\text{Ker } \mathfrak{C}$ in $\mathcal{L}(V)$.

Proof. Suppose $L \in \text{Im } h$ and $\tilde{L} \in \text{Ker } \mathfrak{C}$. A straightforward calculation shows that $\langle L, \tilde{L} \rangle = 2\alpha \tilde{l}$ and this implies the result since $\tilde{l}=0$.

LEMMA 3. $\dim \text{Im } h + \dim \text{Ker } \mathfrak{C} = \dim \mathcal{L}(V)$.

Proof. It follows from Lemma 1 that h is injective and \mathfrak{C} surjective. Hence $\dim \text{Im } h = 1 = \dim \mathcal{L}(V) - \dim \text{Ker } \mathfrak{C}$.

Setting $\mathcal{L}_{conc}(V) = \text{Ker } \mathfrak{C}$ we obtain thus

LEMMA 4. $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}_{conc}(V)$.

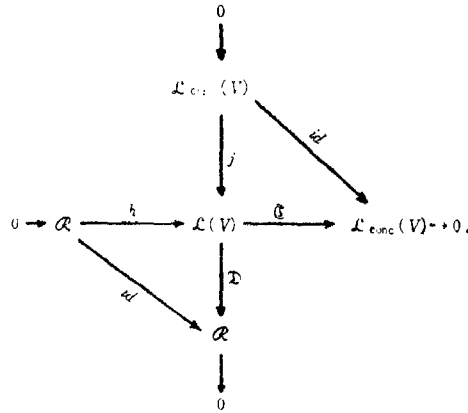
Now let $L \in \mathcal{L}(V)$. Using Lemma 4 we have

$$L = h\alpha + L_{conc}, \quad \alpha \in \mathcal{R}, \quad L_{conc} \in \mathcal{L}_{conc}(V).$$

Define the map $\mathfrak{D} : \mathcal{L}(V) \rightarrow \mathcal{R}$ by $\mathfrak{D}L = \alpha$ and let j be the canonical injection of $\mathcal{L}_{conc}(V)$ in $\mathcal{L}(V)$. Then we have

DECOMPOSITION THEOREM I. *There is a unique linear map $\mathfrak{C} : \mathcal{L}(V) \rightarrow \mathcal{L}_{conc}(V)$, called the concircular map, and a unique linear map $\mathfrak{D} : \mathcal{L}(V) \rightarrow \mathcal{R}$,*

called the deviation map, such that the following commutative diagram with two exact sequences holds:



Moreover, the decomposition $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}_{conc}(V)$ is orthogonal and hence the concircular map \mathfrak{G} is the orthogonal projection of $\mathcal{L}(V)$ onto its subspace $\mathcal{L}_{conc}(V)$.

Further, let $L \in \mathcal{L}(V)$; then

$$\mathfrak{D}L = \frac{l}{n(n-1)},$$

$$\mathfrak{G}L = L - h\mathfrak{D}L = L_{conc} = L - \frac{l}{2n(n-1)}L_{I, I}.$$

Proof. We have $\mathfrak{D}L = \alpha$ and since $l = \text{scalar}(h\alpha) = n(n-1)\alpha$ we find the required formula for α . The rest is now obvious.

DEFINITION. Quasi-curvature tensors belonging to $\mathcal{L}_{conc}(V)$ are called *concircular curvature tensors* and the tensor L_{conc} is called the *concircular tensor* associated with $L \in \mathcal{L}(V)$.

2. The H-concircular curvature tensor.

Now, let V be a $2n$ -dimensional real vector space with a complex structure J and a Hermitian product g , i. e.

$$J^2 = -I, \quad g(Jx, Jy) = g(x, y)$$

for all $x, y \in V$, I denoting the identity transformation on V . Further, let L be a quasi-curvature tensor over V . L is called a *K-quasi-curvature ten-*

so if L satisfies the Kähler identity, i. e.

$$L \circ J = J \circ L$$

and the first Bianchi identity, i. e.

$$\mathfrak{S}L(x, y)z = 0,$$

where \mathfrak{S} denotes the cyclic sum over x, y and z .

We denote by $\mathcal{L}(V)$ the vector space of all K -quasi-curvature tensors and define \mathfrak{C} as before and the linear map h by

$$h : \mathcal{R} \rightarrow \mathcal{L}(V) : \alpha \mapsto \frac{1}{2} \alpha L^{H, I, I}.$$

In this case $L^{H, I, I}$ is the K -quasi-curvature tensor given by

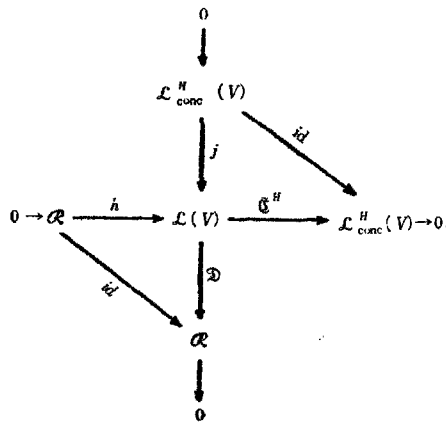
$$L^{H, I, I}(x, y) = 2 \{ x \wedge y + Jx \wedge Jy + 2g(x, Jy)J \}.$$

It is easy to prove that Lemmas 1-4 are still valid if $\mathcal{L}_{conc}(V)$ is replaced by $\mathcal{L}^{H_{conc}}(V) = \text{Ker } \mathfrak{C}$. The first Bianchi identity and the Kähler identity are used to prove the orthogonality of $\text{Im } h$ and $\text{Ker } \mathfrak{C}$. Hence, if $L \in \mathcal{L}(V)$ we have

$$L = h\alpha + L^{H_{conc}}, \quad \alpha \in \mathcal{R}, \quad L^{H_{conc}} \in \mathcal{L}^{H_{conc}}(V),$$

and we define the map \mathfrak{D} as before. Then we get

DECOMPOSITION THEOREM II. *There is a unique linear map $\mathfrak{C}^H : \mathcal{L}(V) \rightarrow \mathcal{L}^{H_{conc}}(V)$, called the H -concircular map, and a unique linear map $\mathfrak{D} : \mathcal{L}(V) \rightarrow \mathcal{R}$, called the deviation map, such that the following commutative diagram with two exact sequences holds:*



Moreover, the decomposition $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}^H_{conc}(V)$ is orthogonal and hence the H -concircular map \mathfrak{C}^H is the orthogonal projection of $\mathcal{L}(V)$ onto its subspace $\mathcal{L}^H_{conc}(V)$.

Further, let $L \in \mathcal{L}(V)$; then

$$\mathcal{D}L = \frac{L}{4n(n+1)},$$

$$\mathfrak{C}^H L = L - h\mathcal{D}L = L^H_{conc} = L - \frac{L}{8n(n+1)} = L^H_{I,I}.$$

DEFINITION. K -quasi-curvature tensors belonging to $\mathcal{L}^H_{conc}(V)$ are called H -concircular curvature tensors and the tensor L^H_{conc} is called the H -concircular tensor associated with $L \in \mathcal{L}(V)$.

3. The projective curvature tensor.

We consider again an n -dimensional ($n > 1$) real vector space with inner product g . A quasi-curvature tensor L is said to be *Ricci symmetric* if the Ricci tensor is symmetric, i. e. $L_R(x, y) = L_R(y, x)$ for all $x, y \in V$.

Let $\text{Hom}_S(V, V)$ denote the vector space of all symmetric endomorphisms of V and suppose $A \in \text{Hom}_S(V, V)$. Then $L_{I,I}A$, i. e.

$$(L_{I,I}A)(x, y) = L_{I,I}(x, y) \circ A,$$

is a Ricci symmetric quasi-curvature tensor.

Denote by $\mathcal{L}(V)$ the vector space of all quasi-curvature tensors which are Ricci symmetric and define the *linear* maps h and \mathfrak{R} by

$$h : \text{Hom}_S(V, V) \rightarrow \mathcal{L}(V) : A \mapsto \frac{1}{2}L_{I,I}A,$$

$$\mathfrak{R} : \mathcal{L}(V) \rightarrow \text{Hom}_S(V, V) : L \mapsto Q.$$

We call \mathfrak{R} the *Ricci contraction map*. Then we have

LEMMA 5. $\mathfrak{R} \circ h$ is an isomorphism.

Proof. It is again sufficient to show the $\mathfrak{R} \circ h$ is injective. Let $A \in \text{Hom}_S(V, V)$ and suppose $\mathfrak{R}hA = 0$. This implies $(n-1)A = 0$ and hence $A = 0$.

LEMMA 6. $\text{Im } h$ is orthogonal to $\text{Ker } \mathfrak{R}$ in $\mathcal{L}(V)$.

Proof. This is an easy calculation using an orthonormal basis of eigenvectors of the symmetric endomorphism A .

LEMMA 7. $\dim \text{Im } h + \dim \text{Ker } \mathfrak{R} = \dim \mathcal{L}(V)$.

Proof. Here we have $\dim \text{Im } h = \dim \text{Hom}_S(V, V) = \dim \mathcal{L}(V) - \dim \text{Ker } \mathfrak{R}$.

Putting $\mathcal{L}_p(V) = \text{Ker } \mathfrak{H}$ we get

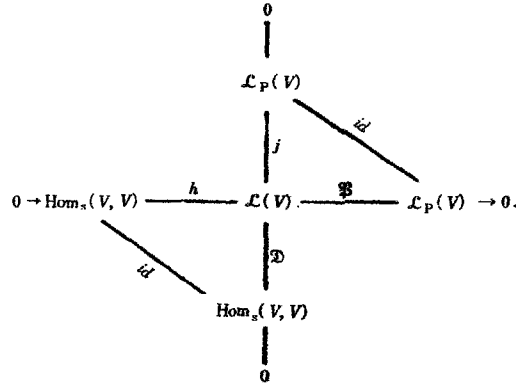
LEMMA 8. $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}_p(V)$.

It follows that for $L \in \mathcal{L}(V)$ we have

$$L = hA + L_p, \quad A \in \text{Hom}_S(V, V), \quad L_p \in \mathcal{L}_p(V).$$

So we define the deviation map in this case by $\mathfrak{D} : \mathcal{L}(V) \rightarrow \text{Hom}_S(V, V) : L \mapsto \mathfrak{D}L = A$. It is now easy to check

DECOMPOSITION THEOREM III. *There is a unique linear map $\mathfrak{F} : \mathcal{L}(V) \rightarrow \mathcal{L}_p(V)$, called the projective map, and a unique linear map $\mathfrak{D} : \mathcal{L}(V) \rightarrow \text{Hom}_S(V, V)$, called the deviation map, such that the following commutative diagram with two exact sequences holds:*



Moreover, the decomposition $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}_p(V)$ is orthogonal and hence the projective map \mathfrak{H} is the orthogonal projection of $\mathcal{L}(V)$ onto its subspace $\mathcal{L}_p(V)$.

Further, let $L \in \mathcal{L}(V)$; then

$$\mathfrak{D}L = \frac{1}{n-1}Q,$$

$$\mathfrak{H}L = L - h\mathfrak{D}L = L_p = L - \frac{1}{2(n-1)}L_{I,I}Q.$$

Proof. We have $\mathfrak{D}L = A$ and since $Q = \text{Ricci}(hA) = (n-1)A$, we obtain the required formulas. The rest follows at once from the lemmas.

DEFINITION. Quasi-curvature tensors belonging to $\mathcal{L}_p(V)$ are called *projective curvature tensors* and the tensor L_p is called the *projective tensor* associated with $L \in \mathcal{L}(V)$.

4. The H -projective curvature tensor.

Suppose V is a $2n$ -dimensional real vector space with a complex structure J and a Hermitian product g . Let $\mathcal{L}(V)$ denote the vector space of all *Ricci symmetric complex linear K -quasi-curvature tensors*. Define \mathfrak{R} as before and let h be the linear map

$$h : \text{Hom}_{CS}(V, V) \rightarrow \mathcal{L}(V) : A \mapsto \frac{1}{2}L^{H,I}A,$$

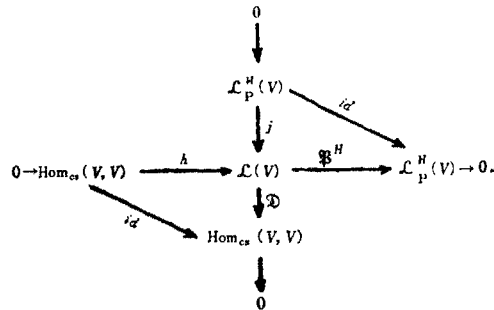
where

$$(L^{H,I}A)(x, y) = (L_{I,I}A)(x, y) + (L_{I,I}A)(Jx, Jy) - 4g(AJx, y)J.$$

$\text{Hom}_{CS}(V, V)$ denotes the space of all symmetric endomorphisms A of V which are *complex linear*, i. e. $A \circ J = J \circ A$.

In the same way as before we obtain with $\mathcal{L}^H_P(V) = \text{Ker } \mathfrak{R}$:

DECOMPOSITION THEOREM IV. *There is a unique linear map $\mathfrak{B}^H = \mathcal{L}(V) \rightarrow \mathcal{L}^H_P(V)$, called the H -projective map, and a unique linear map $\mathfrak{D} : \mathcal{L}(V) \rightarrow \text{Hom}_{CS}(V, V)$, called the deviation map. such that the following commutative diagram with two exact sequences holds:*



Moreover, the decomposition $\mathcal{L}(V) = \text{Im } h \oplus \mathcal{L}^H_P(V)$ is orthogonal and hence the H -projective map \mathfrak{B}^H is the orthogonal projection of $\mathcal{L}(V)$ onto its subspace $\mathcal{L}^H_P(V)$.

Further, let $L \in \mathcal{L}(V)$; then

$$\mathfrak{D}L = \frac{1}{2(n+1)}Q,$$

$$\mathfrak{B}^H_L = L - h\mathfrak{D}L = L^{H_P} = L - \frac{1}{4(n+1)}L^{H_I, I}Q.$$

DEFINITION. K -quasi-curvature tensors belonging to $\mathcal{L}^{H_P}(V)$ are called H -projective curvature tensors and the tensor L^{H_P} is called the H -projective tensor associated with $L \in \mathcal{L}(V)$.

5. Applications to differential geometry.

Let M^n be a Riemannian manifold with metric tensor g and Riemannian connection ∇ . For each point $m \in M$ we may consider quasi-curvature tensors over the tangent space $T_m(M)$ with inner product g_m . A differentiable curvature tensor field L on M is then called a *generalized quasi-curvature tensor field*.

Now, let M^{2n} be an almost Hermitian manifold, that is, the tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the Lie algebra of C^∞ vector fields on M . In the same way as before we may define *generalized K -quasi-curvature tensor fields*.

The Riemann curvature tensor R on M^n provides an example of a quasi-curvature tensor field and if M^{2n} is a Kähler manifold, then R is a K -curvature tensor field.

It is easy to check that in the case $L = R$ all the pointwise constructed tensor fields are the well known tensor fields which occur in differential geometry.

It should be interesting to see under what conditions the curvature tensors are *proper* [4], i. e.

$$\mathfrak{S}(\nabla_X L)(Y, Z) = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$. This is the second Bianchi identity.

6. Remarks.

- a. Let M^{2n} be an almost Hermitian manifold which is not necessarily Kählerian. It is plausible that the given method should also apply to construct H -curvature tensors on some classes of such manifolds. This is done in [8] to define the Bochner curvature tensor on a certain class of almost Hermitian manifolds.
- b. We will show in another paper that, using the same method, a *contact* Bochner curvature tensor can be defined on a class of almost contact metric manifolds. This class includes the well known Sasakian, nearly Sasakian and normal cosymplectic manifolds. It is again plausible that the given method should apply to construct *contact* analogues of the curvature ten-

sors treated in this paper.

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