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CONFORMAL KILLING VECTOR FIELDS IN A *P*-SASAKIAN MANIFOLD

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0. Introduction.

Recently, I. Satō introduced a notion of manifolds with P-Sasakian structure and studied several properties of the manifold which are closely similar to the ones of Sasakian manifolds ([3], [4]).

It is very interesting and important to study some geometrical properties of those vector fields which satisfy certain conditions in a Riemannian manifold.

In this paper, we shall show that a necessary and sufficient condition for a conformal Killing vector field in a P-Sasakian manifold to be concircular is to satisfy the condition $\xi^\nu \nabla_\nu \rho_\lambda = \alpha \eta_\lambda$ (c. f. Theorem 3. 2), and that certain P-Sasakian manifolds satisfy this condition. Furthermore, we shall prove that each of the concircular conformal Killing vector fields in a P-Sasakian manifold is necessarily an infinitesimal automorphism (c. f. Theorem 4. 2).

1. Preliminaries.

An n-dimensional P-Sasakian manifold (or normal paracontact Riemannian manifold) is a Riemannian manifold which admits a unit vector field ξ^λ satisfying

$$\nabla_\mu \eta_\lambda - \nabla_\lambda \eta_\mu = 0, \tag{1.1}$$

$$\nabla_\nu \nabla_\mu \eta_\lambda = (-g_{\nu\lambda} + \eta_\nu \eta_\lambda)\eta_\mu + (-g_{\nu\mu} + \eta_\nu \eta_\mu)\eta_\lambda, \tag{1.2}$$

where $g_{\mu\nu}$ and ∇_μ mean the Riemannian metric and the Levi-Civita covariant differentiation respectively, and $\eta_\lambda \equiv g_{\lambda\mu}\xi^\mu$ ([3]).

Now, if we put

$$\varphi_\mu{}^\lambda = \nabla_\mu \xi^\lambda, \tag{1.3}$$

then we have

$$\eta_\lambda \xi^\lambda = 1, \quad \varphi_\mu{}^\lambda \xi^\mu = 0, \quad \eta_\lambda \varphi_\mu{}^\lambda = 0, \tag{1.4}$$

$$\varphi_\mu{}^\nu \varphi_\nu{}^\lambda = \delta_\mu^\lambda - \eta_\mu \xi^\lambda, \tag{1.5}$$

$$\varphi_{\mu\lambda} = \varphi_{\lambda\mu} \qquad (\varphi_{\mu\lambda} \equiv g_{\lambda\gamma}\varphi_\mu{}^\gamma), \tag{1.6}$$

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(1.7) $$g_{\varepsilon\gamma}\varphi_\mu^\varepsilon\varphi_\lambda^\gamma=g_{\mu\lambda}-\eta_\mu\eta_\lambda.$$

In a P-Sasakian manifold we can prove easily

(1.8) $$R_{\omega\nu\mu}{}^\lambda\eta_\lambda=g_{\omega\mu}\eta_\nu-g_{\nu\mu}\eta_\omega,$$

(1.9) $$R_{\mu\lambda}\xi^\lambda=-(n-1)\eta_\mu,$$

(1.10) $$R_{\mu\varepsilon}\varphi_\lambda^\varepsilon-R_{\mu\varepsilon\gamma\lambda}\varphi^{\varepsilon\gamma}=(n-2)\varphi_{\mu\lambda}-\varphi g_{\mu\lambda}+2\varphi\eta_\mu\eta_\lambda,$$

(1.11) $$R_{\mu\varepsilon}\varphi_\lambda^\varepsilon=R_{\lambda\varepsilon}\varphi_\mu^\varepsilon,$$

where $R_{\omega\nu\mu}{}^\lambda$ and $R_{\mu\lambda}$ are respectively the curvature tensor and the Ricci tensor with respect to $g_{\mu\lambda}$, and φ is defined as $\varphi_{\mu\lambda}g^{\mu\lambda}$ ([2]).

A vector field v^λ is said to be an infinitesimal paracontact transformation if for a certain scalar function α it satisfies

(1.12) $$\mathcal{L}(v)\eta_\lambda=\alpha\eta_\lambda,$$

where $\mathcal{L}(v)$ denotes the Lie differentiation with respect to v^λ. Especially, if α vanishes identically, then v^λ it said to be an infinitesimal strict paracontact transformation.

A vector field v^λ is said to be an infinitesimal automorphism if it leaves invariant three tensors φ_μ^λ, η_λ and $g_{\mu\lambda}$, that is,

(1.13) $$\mathcal{L}(v)\varphi_\mu^\lambda=0,\quad \mathcal{L}(v)\eta_\lambda=0,\quad \mathcal{L}(v)g_{\mu\lambda}=0.$$

A P-Sasakian manifold whose Ricci tensor satisfies

(1.14) $$R_{\mu\lambda}=ag_{\mu\lambda}+b\eta_\mu\eta_\lambda$$

is called an η-Einstein P-Sasakian manifold with the associated functions a and b. For an η-Einstein P-Sasakian manifold we proved

PROPOSITION 1.1. ([5]) *Let M^n $(n>3)$ be a non-Einstein η-Einstein P-Sasakian manifold. A necessary and sufficient condition for the associated functions to be constant is that the vector field ξ^λ be harmonic.*

REMARK 1.1. If M^n is compact orientable, then ξ^λ is harmonic. Thus, in this case, the associated functions are always constant.

REMARK 1.2. If ξ^λ is harmonic, then n is necessarily odd. Thus, in the case of even dimension, the associated functions are never constant.

We know that the eigenvalues of φ_μ^λ are 1, -1, and 0 and that the multiplicity of 0 equals 1. Throughout this paper, we assume that each multiplicity of 1 and -1 does not equal 0, that is, that

(1.15) $$\varphi^2-(n-1)^2\neq 0.$$

In [1], the following propositions were proved:

PROPOSITION 1. 2. *If M^n is Ricci parallel, then it is Einsteinian.*

PROPOSITION 1. 3. *If M^n is symmetric, then it is a manifold of constant curvature* -1.

2. Lemmas.

In this section, we shall prove some lemmas that shall be used in the following sections.

LEMMA 2. 1. *If a scalar function ρ in a P-Sasakian manifold satisfies*

$$\nabla_\mu \rho_\lambda = \sigma g_{\mu\lambda} + \tau \eta_\mu \eta_\lambda, \tag{2.1}$$

σ and τ being certain scalar functions and $\rho_\lambda = \partial_\lambda \rho$, then τ is identically zero.

Proof. Applying the Ricci identity to ρ_λ we have

$$\begin{aligned} -R_{\nu\mu\lambda}{}^{\varepsilon}\rho_\varepsilon = {} & \sigma_\nu g_{\mu\lambda} - \sigma_\mu g_{\nu\lambda} + \tau_\nu \eta_\mu \eta_\lambda - \tau_\mu \eta_\nu \eta_\lambda \\ & + \tau (\eta_\mu \varphi_{\nu\lambda} - \eta_\nu \varphi_{\mu\lambda}), \end{aligned} \tag{2.2}$$

where we put $\sigma_\nu = \partial_\nu \sigma$ and $\tau_\nu = \partial_\nu \tau$.
Transvecting (2. 2) with $\xi^\nu g^{\mu\lambda}$, we have

$$(n-1)\xi^\varepsilon \rho_\varepsilon = (n-1)\sigma_\varepsilon \xi^\varepsilon - \varphi\tau. \tag{2.3}$$

On the other hand, transvecting (2. 2) with $\xi^\nu \varphi^{\mu\lambda}$ and taking account of (1. 10), we have

$$\varphi \xi^\varepsilon \rho_\varepsilon = \varphi \xi^\varepsilon \sigma_\varepsilon + (n-1)\tau. \tag{2.4}$$

By virtue of (2. 3) and (2. 4), we have

$$\{\varphi^2 - (n-1)^2\}\tau = 0. \tag{2.5}$$

(1. 15) and (2. 5) complete the proof of Lemma 2. 1.

LEMMA 2. 2. *If $\varphi_\mu{}^\lambda$ in a P-Sasakian manifold satisfies*

$$\mathcal{L}(v)\varphi_\mu{}^\lambda = \rho \varphi_\mu{}^\lambda + \sigma(\delta_\mu^\lambda - \eta_\mu \xi^\lambda) \tag{2.6}$$

for a vector field v^λ and certain scalar functions ρ and σ, then ρ and σ are identically zero.

Proof. From (2. 6) and $\mathcal{L}(v)\varphi = 0$, we have

$$\rho\varphi + (n-1)\sigma = 0. \tag{2.7}$$

Lie differentiating (1. 5) with respect to v^λ and using (2. 6), we have

$$2\rho(\delta_\mu^\lambda - \eta_\mu \xi^\lambda) + 2\sigma \varphi_\mu{}^\lambda = -(\mathcal{L}(v)\eta_\mu)\xi^\lambda - \eta_\mu \mathcal{L}(v)\xi^\lambda,$$

from which

$$(n-1)\varphi+\varphi\sigma=0. \tag{2.8}$$

By virtue of (2.7) and (2.8), we get

$$\{\varphi^2-(n-1)^2\}\rho=0. \tag{2.9}$$

(1.15), (2.7) and (2.9) complete the proof of Lemma 2.2.

3. Conformal Killing vector fields in a *P*-Sasakian manifold.

Let v^λ be a conformal Killing vector field with the associated function ρ in a *P*-Sasakian manifold M^n. Then, by definition, we have

$$\mathcal{L}(v)g_{\mu\lambda}=2\rho g_{\mu\lambda}. \tag{3.1}$$

Especially, if ρ satisfies

$$\nabla_\mu\nabla_\lambda\rho=\alpha g_{\mu\lambda}, \tag{3.2}$$

α being a certain scalar function, then v^λ is said to be concircular. Moreover, if ρ is constant, then v^λ is said to be homothetic.

From (3.1), we get

$$\mathcal{L}(v)\{{}_\nu{}^\lambda{}_\mu\}=\delta_\nu^\lambda\rho_\mu+\delta_\mu^\lambda\rho_\nu-\rho^\lambda g_{\nu\mu}, \tag{3.3}$$

$$\mathcal{L}(v)R_{\omega\nu\mu}{}^\lambda=-\delta_\omega^\lambda\nabla_\nu\rho_\mu+\delta_\nu^\lambda\nabla_\omega\rho_\mu-(\nabla_\omega\rho^\lambda)g_{\nu\mu}+(\nabla_\nu\rho^\lambda)g_{\omega\mu}, \tag{3.4}$$

where $\{{}_\nu{}^\lambda{}_\mu\}$ are the Christoffel symbols with respect to $g_{\mu\lambda}$ and $\rho_\lambda\equiv\partial_\lambda\rho$ ([6]).

First, let us prove the following

LEMMA 3.1 *The associated function ρ of a conformal Killing vector field v^λ in M^n satisfies*

$$\xi^\lambda\mathcal{L}(v)\eta_\lambda=\rho. \tag{3.5}$$

Proof. Transvecting (3.1) with $\xi^\mu\xi^\lambda$, we have

$$\xi^\mu\xi^\lambda\mathcal{L}(v)g_{\mu\lambda}=2\rho. \tag{3.6}$$

On the other hand, we have

$$\xi^\mu\xi^\lambda\mathcal{L}(v)g_{\mu\lambda}=\xi^\mu(\mathcal{L}(v)\eta_\mu-g_{\mu\lambda}\mathcal{L}(v)\xi^\lambda)=2\xi^\mu\mathcal{L}(v)\eta_\mu.$$

From (3.6) and the above equation, Lemma 3.1 is proved.

Taking account of Lemmas 2.1 and 3.1, we shall prove the following

THEOREM 3.2. *Let v^λ be a conformal Killing vector field with the associated function ρ in M^n. In order for v^λ to be concircular it is necessary and sufficient that ρ satisfy*

$$\xi^\tau\nabla_\tau\rho_\lambda=\alpha\eta_\lambda,\quad(\alpha=\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau). \tag{3.7}$$

Proof. By virtue of (3. 2), it is clear that the associated function ρ of a concircular conformal Killing vector field satisfies (3. 7). Thus, it is sufficient that a conformal Killing vector field v^λ whose associated function ρ satisfies (3. 7) be concircular.

We assume that the associated function ρ of a conformal Killing vector field v^λ satisfies (3. 7).
Transvecting (3. 4) with η_λ, we have

$$(3.8)\qquad (\mathcal{L}(v) R_{\omega\nu\mu}{}^\lambda)\eta_\lambda = -\eta_\omega \nabla_\nu \rho_\mu + \eta_\nu \nabla_\omega \rho_\mu - \eta_\lambda (\nabla_\omega \rho^\lambda) g_{\omega\mu} + \eta_\lambda (\nabla_\nu \rho^\lambda) g_{\omega\mu}.$$

By virtue of (1. 8) and (3. 1), we have

$$(3.9)\qquad \mathcal{L}(v)(R_{\omega\nu\mu}{}^\lambda \eta_\lambda) = 2\rho(g_{\omega\mu}\eta_\nu - g_{\nu\mu}\eta_\omega) + g_{\omega\mu}\mathcal{L}(v)\eta_\nu - g_{\nu\mu}\mathcal{L}(v)\eta_\omega.$$

Substituting (3. 8) and (3. 9) in the ideaity

$$\mathcal{L}(v)(R_{\omega\nu\mu}{}^\lambda \eta_\lambda) = (\mathcal{L}(v) R_{\omega\nu\mu}{}^\lambda)\eta_\lambda + R_{\omega\nu\mu}{}^\lambda \mathcal{L}(v)\eta_\lambda,$$

we obtain

$$(3.10)\qquad R_{\omega\nu\mu}{}^\lambda \mathcal{L}(v)\eta_\lambda = \eta_\omega \nabla_\nu \rho_\mu - \eta_\nu \nabla_\omega \rho_\mu + \xi^\lambda (\nabla_\omega \rho_\lambda) g_{\nu\mu} - \xi^\lambda (\nabla_\nu \rho_\lambda) g_{\omega\mu} + 2\rho(g_{\omega\mu}\eta_\nu - g_{\nu\mu}\eta_\omega) + g_{\omega\mu}\mathcal{L}(v)\eta_\nu - g_{\nu\mu}\mathcal{L}(v)\eta_\omega.$$

Transvecting (3. 10) with ξ^ω and taking account of (3. 7) and Lemma 3. 1, we find

$$(3.11)\qquad \nabla_\nu \rho_\mu = (2\rho - \alpha) g_{\nu\mu} - 2(\rho - \alpha)\eta_\nu \eta_\mu.$$

Thus we have from Lemma 2. 1 and (3. 11)

$$(3.12)\qquad \nabla_\nu \rho_\mu = \rho g_{\nu\mu}.$$

This completes the proof of Theorem 3. 2.

Contracting $g^{\nu\mu}$ to (3. 10), we have

$$(3.13)\qquad R_\omega{}^\lambda \mathcal{L}(v)\eta_\lambda = (\nabla_\gamma \rho^\gamma)\eta_\omega - 2(n-1)\rho\eta_\omega + (n-2)\xi^\gamma \nabla_\omega \rho_\gamma - (n-1)\mathcal{L}(v)\eta_\omega.$$

Transvecting (3. 10) with $\varphi^{\nu\mu}\varphi_\tau{}^\omega$ and taking account of (1. 10), (1. 11) and Lemma 3. 1, we have

$$(3.14)\qquad R_\tau{}^\lambda \mathcal{L}(v)\eta_\lambda = (n-1)\mathcal{L}(v)\eta_\tau - 2\varphi_\tau{}^\omega \mathcal{L}(v)\eta_\omega - 2(n-1)\rho\eta_\tau + \varphi\xi^\lambda \varphi_\tau{}^\omega \nabla_\omega \rho_\lambda - \xi^\lambda \nabla_\tau \rho_\lambda + (\xi^\varepsilon \xi^\gamma \nabla_\varepsilon \xi_\gamma)\eta_\tau.$$

Thus we have from (3.13) and (3.14)

$$2(n-1)\mathcal{L}(v)\eta_\lambda-2\varphi\varphi_\lambda{}^\tau\mathcal{L}(v)\eta_\tau+(\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau-\nabla_\tau\rho^\tau)\eta_\lambda$$
$$-(n-1)\xi^\tau\nabla_\lambda\rho_\tau+\varphi\xi^\tau\varphi_\lambda{}^\varepsilon\nabla_\varepsilon\rho_\tau=0. \tag{3.15}$$

Comparing (3.15) with the equation that transvects (3.15) with $\varphi_\mu{}^\lambda$, we find

$$\{(n-1)^2-\varphi^2\}(2\mathcal{L}(v)\eta_\lambda-\xi^\tau\nabla_\lambda\rho_\tau)+\{(n-1)(\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau$$
$$-\nabla_\tau\rho^\tau)+\varphi^2(2\rho-\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau)\}\eta_\lambda=0.$$

By virtue of (1.15), Lemma 3.1, and the above equation, we obtain

$$2\mathcal{L}(v)\eta_\lambda=2\rho\eta_\lambda+\xi^\tau\nabla_\lambda\rho_\tau-(\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau)\eta_\lambda. \tag{3.16}$$

Next, if we assume that M^n is η-Einsteinian, then we have from (1.14) and (3.13)

$$\{a+(n-1)\}\mathcal{L}(v)\eta_\lambda=\{\nabla_\tau\rho^\tau-2(n-1)\rho-b\rho\}\eta_\lambda$$
$$+(n-2)\xi^\tau\nabla_\lambda\rho_\tau.$$

Substituting (3.16) in the above equation, we get

$$\{a-(n-3)\}\xi^\tau\nabla_\lambda\rho_\tau=\{2\nabla_\tau\rho^\tau-4(n-1)\rho-b(\xi^\varepsilon\xi^\tau\nabla_\varepsilon\rho_\tau)\}\eta_\lambda. \tag{3.17}$$

Thus we have

COROLLARY 3.3. *Each of the conformal Killing vector fields in an η-Einstein P-Sasakian manifold satisfying $a\neq n-3$ is concircular.*

From Remark 1.2, we have

COROLLARY 3.4. *Each of the conformal Killing vector fields in an even dimensional η-Einstein P-Sasakian manifold is concircular.*

Especially, if $a\neq-(n-1)$ in (1.14), then the manifold is Einsteinian. Thus we have from Corollary 3.3

COROLLARY 3.5. *Each of the conformal Killing vector fields in an Einstein P-Sasakian manifold is concircular.*

If M^n is a manifold of constant curvature, then its curvature is always equal to -1 and M^n is Einsteinian. Thus we have from the above corollary

COROLLARY 3.6. *Each of the conformal Killing vector fields in a P-Sasakian manifold of constant curvature is concircular.*

Propositions 1.2, 1.3 and Corollaries 3.5, 3.6 denote the following

COROLLARY 3.7. *Each of the conformal Killing vector fields in a symmetric or a Ricci parallel P-Sasakian manifold is concircular.*

4. An infinitesimal automorphism in a P-Sasakian manifold.

In this section, we shall consider the relation between a conformal Killing vector field and an infinitesimal automorphism in a P-Sasakian manifold M^n.

First of all, we shall prove the following

THEOREM 4. 1. *A necessary and sufficient condition for a conformal Killing vector field in M^n to be an infinitesimal paracontact transformation is that the vector field be concircular.*

Proof. If a conformal Killing vector field v^λ with the associated function ρ is an infinitesimal paracontact transformation, then we have

$$\mathcal{L}(v)\eta_\lambda = f\eta_\lambda$$

for a certain scalar function f. Substituting the above equation in (3. 16) and transvecting this equation with $\varphi_\mu{}^\lambda$, we have

$$\varphi_\mu{}^\lambda \eta^\nu \nabla_\lambda \rho_\nu = 0,$$

from which

$$\xi^\nu \nabla_\mu \rho_\nu = (\xi^\varepsilon \xi^\nu \nabla_\varepsilon \rho_\nu)\eta_\mu. \tag{4. 1}$$

Hence, by virtue of Theorem 3. 2 and (4. 1), v^λ is concircular. Conversely, if a vector field v^λ is a concircular conformal Killing vector field with the associated function ρ, then we have from Theorem 3. 2 and (3. 16)

$$\mathcal{L}(v)\eta_\lambda = \rho\eta_\lambda, \tag{4. 2}$$

which means that v^λ is an infinitesimal paracontact transformation.

Next, we shall prove

THEOREM 4. 2. *A concircular conformal Killing vector field in M^n is necessarily an infinitesimal automorphism.*

Proof. For a concircular conformal Killing vector field v^λ with the associated function ρ, we have from (3. 1) and (4. 2)

$$\mathcal{L}(v)\xi^\lambda = -\rho\xi^\lambda. \tag{4. 3}$$

On the other hand, in a Riemannian manifold, the following identity holds good ([6]):

$$\mathcal{L}(u)\nabla_\mu w^\lambda - \nabla_\mu \mathcal{L}(u) w^\lambda = (\mathcal{L}(u)\left\{{}_{\mu}{}^{\lambda}{}_{\varepsilon}\right\}) w^\varepsilon \tag{4. 4}$$

for any vector fields u^λ and w^λ.
By virtue of (1. 3), (3. 3), (4. 3), (4. 4), and the equation $\rho^\lambda = (\rho_\varepsilon \xi^\varepsilon)\xi^\lambda$, we have

(4.5) $$\mathcal{L}(v)\varphi_\mu{}^\lambda = -\rho\varphi_\mu{}^\lambda + (\rho_\varepsilon\xi^\varepsilon)(\delta_\mu^\lambda - \eta_\mu\xi^\lambda).$$

Taking account of Lemma 2.2 and (4.5), we get

(4.6) $$\rho = 0.$$

(3.1), (4.2), (4.5), and (4.6) complete the proof of Theorem 4.2.

From the corollaries and the above theorem, we have the followings

COROLLARY 4.3. *Each of the homothetic conformal Killing vector fields in a P-Sasakian manifold is an infinitesimal automorphism.*

COROLLARY 4.4. *Each of the conformal Killing vector fields in an η-Einstein P-Sasakian manifold satisfying $a \neq n-3$ is an infinitesimal automorphism.*

COROLLARY 4.5. *Each of the conformal Killing vector fields in an Einstein P-Sasakian manifold is an infinitesimal automorphism.*

COROLLARY 4.6. *Each of the conformal Killing vector fields in a P-Sasakian manifold of constant curvature is an infinitesimal automorphism.*

COROLLARY 4.7. *Each of the conformal Killing vector fields in a symmetric or a Ricci parellel P-Sasakian manifolds is an infinitesimal automorphism .*

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