J. Korean Math. Soc.
Vol. 14, No. 1, 1977.

GENERALIZED f-CONTRACTIONS AND FIXED POINT THEOREMS

BY SEHIE PARK AND KISOO PARK

1. Introduction. The Banach contraction principle has had numerous generalizations. The following is due to C. S. Wong [8].

THEOREM A. *Let T be a selfmap of a complete metric space (X, d). Then T has a fixed point if there exist selfmaps $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ on $[0, \infty)$ such that*

(1) $\sum_{i=1}^{5}\alpha_i(t)<t$ *for* $t>0$,

(2) *each α_i is upper-semicontinuous from the right,*

(3) $d(Tx, Ty)\leqslant a_1 d(x,y)+a_2 d(x, Tx)+a_3 d(y, Ty)+a_4 d(x, Ty)+a_5 d(y, Tx)$

for all pairs of distinct x, y in X, where $a_i=\alpha_i(d(x,y))/d(x,y)$.

This theorem also generalizes some results of D. W. Boyd and J. S. W. Wong [1] and of F. E. Browder [2].

We note that a fixed point of a map $g: X\to X$ is a common fixed point of g and the identity map 1_X of X. Motivated by this fact, G. Jungck [5] and S. Park [6], [7] obtained some results on fixed points by replacing 1_X by a continuous map $f: X\to X$. In fact, in [5], G. Jungck obtained the following useful extension of the Banach contraction principle.

THEOREM B. *A continuous selfmap f of a complete metric space (X, d) has a fixed point iff there exists $\alpha\in[0,1)$ and a map $g: X\to X$ which commutes with f and satisfies $gX\subset fX$ and $d(gx, gy)\leqslant \alpha d(fx, fy)$ for all $x, y\in X$. Indeed, f and g have a unique common fixed point ζ.*

In the present paper, we obtain a combined form of Theorems A and B and related results. Consequently, our results extend those in [1], [2], [5] and [8]. Actually our results will be stated for metric spaces more general than complete ones.

2. Generalized f-contractions. Let f be a continuous selfmap of a metric space (X, d), and C_f denote the family of maps $g: X\to X$ such that $gX\subset fX$ and $gf=fg$. Note that C_f is not emptly since f itself belongs to C_f.

Received by the editors Apr. 28, 1977.

DEFINITION. Given $x \in X$ and a map $g \in C_f$, an *f-iteration of x under g* is a sequence $\{fx_n\}_{n=1}^{\infty}$ given inductively by the rule $fx_n = gx_{n-1}$ for all $n \geqslant 1$, where $x_0 = x$.

Note that given $x \in X$, its f-iteration is not unique, however, in case $f = 1_X$, an f-iteration of x under g is reduced to the (Picard) sequence of iterates for g.

PROPOSITION 2. 1. *If $x \in X$ has an f-iteration under $g \in C_f$ whose limit is $\eta \in X$, then each $fx_i \in X$ $(i \geqslant 1)$ has an f-iteration whose limit is $f\eta$.*

Proof. Since $ffx_n = fgx_{n-1} = gfx_{n-1}$, $\{ffx_n\}_{n>i}$ is an f-iteration of fx_i under g. From $fx_n \to \eta$ and the continuity of f, we have $ffx_n \to f\eta$.

DEFINITION. A map g in C_f is called a *generalized f-contraction* in the sense of C. S. Wong if there exist functions $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ of $(0, \infty)$ into $[0, \infty)$ such that

(a) each α_i is upper-semicontinuous from the right,

(b) $\sum_{i=1}^{5} \alpha_i(t) < t$, $t > 0$,

(c) for any $x, y \in X$, $fx \neq fy$, we have

$d(gx, gy) \leqslant a_1 d(fx, fy) + a_2 d(fx, gx) + a_3 d(fy, gy) + a_4 d(fx, gy) + a_5 (fy, gx)$

where $a_i = \alpha_i(d(fx, fy))/d(fx, fy)$.

Note that if $f = 1_X$, the identity map of X, then we get a generalized contraction of C. S. Wong [8].

For a generalized f-contraction, the limit of an f-iteration of x under g depends only on $x \in X$ if exists.

PROPOSITION 2. 2. *Let $g \in C_f$ be a generalized f-contraction. For any $x \in X$, if an f-iteration of x under g has the limit η, so does every convergent f-iteration of x under g.*

Proof. Suppose an f-iteration $\{fx_n\} = \{gx_{n-1}\}_{n=1}^{\infty}$ of $x = x_0 \in X$ has limit $\eta \in X$. Let $\{fy_n\} = \{gy_{n-1}\}_{n=1}^{\infty}$ be another convergent f-iteration of $x = y_0$ under g. Then there exists an $i \geqslant 1$ such that $fx_i \neq fy_i$. Hence, we have

$$\begin{aligned} d(\eta, gy_i) &\leqslant d(\eta, gx_i) + d(gx_i, gy_i) \\ &\leqslant d(\eta, gx_i) + a_1 d(fx_i, fy_i) + a_2 d(fx_i, gx_i) \\ &\quad + a_3 d(fy_i, gy_i) + a_4 d(fx_i, gy_i) + a_5 d(fy_i, gx_i) \end{aligned}$$

and, hence,

$$d(\eta, gy_i) \leqslant a_1 d(fx_i, gy_{i-1}) + a_4 d(fx_i, gy_i) + a_5 d(fx_{i+1}, gy_{i-1}) + o(i)$$

where $\{o(i)\}$ converges to zero. Since $a_1 + a_4 + a_5 < 1$, by letting $i \to \infty$, we have $gy_i \to \eta$.

THEOREM 2.3. *For any generalized f-contraction $g \in C_f$ and for any $x \in X$, there is a Cauchy f-iteration of x under g.*

Proof. Given $x \in X$, we choose an f-iteration $\{fx_n\} = \{gx_{n-1}\}$ of $x = x_0$ under g as follows: If we have $fx_{i+1} = fx_i$ for some i, then we can choose fx_{i+j} by fx_{i+1} for all $j \geqslant 1$ and obtain a Cauchy f-iteration. So we may assume that $d(fx_{n+1}, fx_n) > 0$ for each n. Then we have

$$\begin{aligned} d(fx_{n+1}, fx_n) &= d(gx_n, gx_{n-1}) \\ &\leqslant a_1 d(fx_n, fx_{n-1}) + a_2 d(fx_n, gx_n) + a_3 d(fx_{n-1}, gx_{n-1}) \\ &\quad + a_4 d(fx_n, gx_{n-1}) + a_5 d(fx_{n-1}, gx_n) \\ &\leqslant (a_1 + a_3 + a_5) d(fx_n, fx_{n-1}) + (a_2 + a_5) d(fx_n, fx_{n+1}) \end{aligned}$$

and, hence,

$$d(fx_{n+1}, fx_n) \leqslant ((a_1 + a_3 + a_5)/(1 - a_2 - a_5)) d(fx_n, fx_{n-1}).$$

By symmetry of x, y in (c), we may assume that $\alpha_5 = \alpha_4$. So if we define a function

$$\alpha(t) = [(\alpha_1(t) + \alpha_3(t) + \alpha_5(t))/(t - \alpha_2(t) - \alpha_4(t))]t, \quad t > 0,$$

then

$$d(fx_{n+1}, fx_n) \leqslant \alpha(d(fx_n, fx_{n-1}))$$

for all $n \geqslant 1$. Since $\alpha(t) < t$ for $t > 0$ by (b), $\{d(fx_{n+1}, fx_n)\}$ is decreasing and converges to some $s \in [0, \infty)$. If $s > 0$, then

$$s = \lim_{n\to\infty} d(fx_{n+1}, fx_n) \leqslant \limsup_{n\to\infty} \alpha(d(fx_n, fx_{n-1})).$$

Since α is upper-semicontinuous from the right by (a), we have $s \leqslant \alpha(s)$, a contradiction. So $s = 0$. Now we prove that $\{fx_n\}$ is Cauchy. Suppose not. Then there exist $r > 0$ and sequences $\{p(n)\}$, $\{q(n)\}$ such that for each n,

$$p(n) > q(n) > n, \quad d(fx_{p(n)}, fx_{q(n)}) \geqslant r$$

and by the well-ordering principle

$$d(fx_{p(n)-1}, fx_{q(n)}) < r.$$

Then

$$\begin{aligned} r &\leqslant d(fx_{p(n)}, fx_{q(n)}) \\ &\leqslant d(fx_{p(n)-1}, fx_{q(n)}) + d(fx_{p(n)}, fx_{p(n)-1}) \\ &\leqslant r + o(n) \end{aligned}$$

where $o(n)$ converges to 0, hence, $\{d(fx_{p(n)}, fx_{q(n)})\}$

converges to r from the right. By (c),

$$d(fx_{p(n)}, fx_{q(n)})d(gx_{p(n)}, gx_{q(n)})$$
$$\leqslant d(fx_{p(n)}, fx_{q(n)})[a_1 d(fx_{p(n)}, fx_{q(n)})$$
$$+a_2 d(fx_{p(n)}, fx_{p(n)+1}) + a_3 d(fx_{q(n)}, fx_{q(n)+1})$$
$$+a_4 d(fx_{p(n)}, fx_{q(n)+1}) + a_5 d(fx_{q(n)}, fx_{p(n)+1})].$$

So by letting $n \to \infty$, we obtain

$$r^2 \leqslant r(a_1 r + a_4 r + a_5 r),$$

a contradiction to (b). Hence $\{fx_n\}$ is Cauchy.

Given a continuous selfmap f of (X, d), we consider a condition on X somewhat more general than completeness.

DEFINITION. Given g in C_f, X is said to be *g-orbitally complete* w. r. t. f if, for any $x \in X$, every Cauchy subsequence of an arbitrary f-iteration $\{fx_n\}_{n=1}^{\infty}$ of x under g converges in X. If X is g-orbitally complete for any $g \in C_f$, then X is said to be *f-complete.*

The g-orbital completeness w. r. t. 1_X is just the g-orbital completeness of Ciric [3], [4].

Clearly, every complete metric space X is f-complete for any f. However, the converse is not true. For example, it may happen that X is not complete, but fX is complete. Then X is f-complete. Note also that every metric space X is g-orbitally complete w. r. t. f if gX is complete.

Combining Proposition 2.2 and Theorem 2.3, we have

COROLLARY 2.4. *Let f be a continuous selfmap of X. If $g \in C_f$ is a generalized f-contraction such that X is g-orbitally complete w. r. t. f, then any $x \in X$ has a convergent f-iteration under g and its limit depends only on x.*

3. Fixed Point Theorems.

Now we have the following main theorem:

THEOREM 3.1. *A continuous selfmap f of a metric space X has a fixed point iff there is a generalized f-contraction g in C_f such that X is g-orbitally complete w. r. t. f. Indeed,*

(1) f and g have a unique common fixed point $\eta \in X$, and

(2) for any $x_0 \in X$ and any f-iteration $\{fx_n\}$ of x_0 under g, we have $\lim_n gfx_n = \eta$.

Proof. Necessity. Suppose that $f\eta = \eta$ for some $\eta \in X$. Define $g: X \to X$ by $gx = \eta$ for all $x \in X$. Then clearly $g \in C_f$. Define $\alpha_1 = (0, \infty) \to [0, \infty)$ by $\alpha_1(t) = \lambda t$ for any $\lambda \in (0, 1)$ and $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. Then (a), (b), and (c) clearly follow.

Sufficiency. Let g be a generalized f-contraction in C_f such that X is g-

orbitally complete w. r. t. f. For any $x=x_0\in X$, we have a convergent f-iteration $\{fx_n\}=\{gx_{n-1}\}$, $n\geqslant 1$, of x, by Corollary 2.4. Let ζ be its limit. Now we show that $f\zeta=g\zeta$. If $fx_{i+1}=fx_i$ for some i, then we could have chosen $fx_{i+1}=fx_{i+2}=\cdots=\zeta$ as in the proof of Theorem 2.3. Hence, $gfx_{i+1}=gfx_{i+2}=\cdots=g\zeta$. Since $gfx_{i+1}=fgx_{i+1}=ffx_{i+2}$, we know that $\{gfx_n\}$ is a tail of an f-iteration of fx_i and, hence, has the limit $f\zeta$, by Proposition 2.1. Therefore, we have $f\zeta=g\zeta$. Suppose that $fx_{n+1}\neq fx_n$ for all $n\geqslant 1$. If $ffx_i\neq f\zeta$ for some $i\geqslant 1$, then

$$\begin{aligned} d(f\zeta,g\zeta)&\leqslant d(f\zeta,fgx_i)+d(gfx_i,g\zeta)\\ &\leqslant d(f\zeta,fgx_i)+a_1d(ffx_i,f\zeta)+a_2d(ffx_i,gfx_i)\\ &\quad+a_3d(f\zeta,g\zeta)+a_4d(ffx_i,g\zeta)+a_5d(f\zeta,gfx_i)\end{aligned}$$

where $a_i=\alpha_i(d(ffx_i,f\zeta))/d(ffx_i,f\zeta)$. Since $ffx_i=fgx_{i-1}=gfx_{i-1}$, letting $i\to\infty$, we obtain

$$d(f\zeta,g\zeta)\leqslant(a_3+a_4)\,d(f\zeta,g\zeta),$$

whence we have $f\zeta=g\zeta$. If $ffx_n=f\zeta$ for all $n\geqslant 1$, we have also $f\zeta=g\zeta$. Suppose now $f\zeta\neq fg\zeta$. Then

$$\begin{aligned} d(f\zeta,fg\zeta)&=d(g\zeta,g^2\zeta)\\ &\leqslant a_1d(f\zeta,fg\zeta)+a_2d(f\zeta,g\zeta)+a_3d(fg\zeta,g^2\zeta)\\ &\quad+a_4d(f\zeta,g^2\zeta)+a_4d(f\zeta,g^2\zeta)+a_5d(fg\zeta,f\zeta)\\ &=(a_1+a_4+a_5)\,d(f\zeta,fg\zeta).\end{aligned}$$

Since $a_1+a_4+a_5<1$, this leads a contradiction. Hence, we must have $f\zeta=fg\zeta$, which shows that $f\zeta=g\zeta$ is a common fixed point of f and g. If f and g have two common fixed point α,β in X, $\alpha\neq\beta$, then

$$\begin{aligned} d(\alpha,\beta)&=d(g\alpha,g\beta)\\ &\leqslant a_1d(\alpha,\beta)+a_4d(\alpha,\beta)+a_5d(\alpha,\beta)<d(\alpha,\beta),\end{aligned}$$

which leads a contradiction. Hence f and g have a unique common fixed point.

In case $f=1_X$, a generalized contraction g has a unique fixed point by Theorem 3.1. This is just Theorem 1 of C. S. Wong [8], which is reduced to a result of Boyd-Wong [1]. If $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and α_1 is a constant in [0, 1), Theorem 3.1 is reduced to a result of Jungck [5]. In fact, a number of known fixed point theorems are consequences of Theorem 3.1.

In Theorem 3.1, it can be proved that if X is compact and if "$<$" in (b) is interchanged with "$\leqslant$" in (c), then it is still true.

COROLLARY 3.2. *Let f be a continuous selfmap of a complete metric space*

(X, d). If $g\in C_f$ and if there is a positive integer k such that g^k is a generalized f-contraction, then f and g have a unique common fixed point.

Proof. Clearly $g^k\in C_f$ and by Theorem 3.1, there is a unique $\eta\in X$ such that $\eta=f\eta=g^k\eta$. But then, since f and g commute, we can write $g\eta=f(g\eta)=g^k(g\eta)$, which says that $g\eta$ is also a common fixed point of f and g^k. The uniqueness implies $\eta=g\eta=f\eta$.

THEOREM 3.3. *Let f be a continuous selfmap of a metric space X and g be a generalized f-contraction in C_f such that X is g-orbitally complete w.r.t. f as in* Theorem 3.1. *Suppose further that each α_i is increasing. Then*

(i) *$d(gfx_n, \eta)\leqslant\alpha^n(d(gfx, \eta))$ for all $n\geqslant 0$ where η is the common fixed point of f and g, $\alpha(0)=0$, and for $t>0$,*

$$\alpha(t)=[(2\alpha_1(t)+\textstyle\sum_{i=2}^{5}\alpha_i(t))/(2t-\sum_{i=2}^{5}\alpha_i(t))]t;$$

(ii) *α is increasing, continuous from the right and for any $t\in[0, \infty)$, $\{\alpha^n(t)\}$ converges to 0.*

Hence, $\{gfx_n\}$ converges uniformly to the common fixed point of f and g on any bounded subset of X.

Proof. (i) Let $x\in X$ be such that $gfx\neq\eta$ and $b_n=d(gfx_n, \eta)$ for $n\geqslant 0$. By (c), we have

$$\begin{aligned}b_0b_1=b_0d(gfx_1, \eta)&=b_0d(g^2x, g\eta)\\ &\leqslant\alpha_1(b_0)d(fgx, \eta)+\alpha_2(b_0)d(fgx, g^2x)\\ &\qquad+\alpha_4(b_0)d(fgx, \eta)+\alpha_5(b_0)d(\eta, g^2x)\\ &\leqslant\alpha_1(b_0)b_0+\alpha_2(b_0)(b_0+b_1)+\alpha_4(b_0)b_0+\alpha_5(b_0)b_1\end{aligned}$$

and, hence,

$$b_1\leqslant[(\alpha_1(b_0)+\alpha_2(b_0)+\alpha_4(b_0))/(b_0-\alpha_2(b_0)-\alpha_5(b_0))]b_0.$$

Similarly

$$b_2\leqslant[(\alpha_1(b_1)+\alpha_2(b_1)+\alpha_4(b_1))/(b_1-\alpha_2(b_1)-\alpha_5(b_1))]b_1.$$

Because of the symmetry of x, y, (c) still holds if we replace $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ respectively by

$$\alpha_1,\ (\alpha_2+\alpha_3)/2,\ (\alpha_2+\alpha_3)/2,\ (\alpha_4+\alpha_5)/2,\ (\alpha_4+\alpha_5)/2.$$

Thus $b_1\leqslant\alpha(b_0)$, $b_2\leqslant\alpha(b_1)$, and by induction $b_{n+1}\leqslant\alpha(b_n)$, $n\geqslant 0$. Since α is increasing, by induction we have

$$d(gfx_n, \eta)=b_n\leqslant\alpha^n(b_0)=\alpha^n(d(gfx, x)),\quad n\geqslant 0.$$

(ii) Each α_i is increasing and continuous from the right, so is α. Let $t>0$. By (b), $\alpha(t)<t$. So $\{\alpha^n(t)\}$ is decreasing and converges to some $t_0\in$

$[0, \infty)$. Suppose $t_0>0$. Then by the right continuity of α,

$$t_0=\lim_{n\to\infty}\alpha^{n+1}(t)\leqslant\alpha(\lim_{n\to\infty}\alpha^n(t))=\alpha(t_0),$$

which is a contradiction to $\alpha(t)<t$ for $t>0$. Hence, $t_0=0$.

In case $f=1_X$, Theorem 3.3 is due to C. S. Wong [8]. Note that our proof is a slight modification of his. When $f=1_X$, $\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and X is bounded, Theorem 3.3 is reduced to a result of F. E. Browder [2, Theorem 1].

References

[1] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc. **20** (1969), 458-464. MR **39** #916.

[2] F. E. Browder, *On the convergence of successive approximation for nonlinear functional equations*, Nederl. Akad. Wetensch. Proc. Ser. A **71**=Indag. Math. **30** (1968), 27-35. MR **37** #5743.

[3] Lj. B. Ciric, *Generalized contractions and fixed-point theorems*, Publ. Inst. Math. (Beograd) (N. S.) **12**(**26**) (1971), 19-26. MR **46** #8203.

[4] ________, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267-273. MR **50** #8484.

[5] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83** (1976), 261-263.

[6] Sehie Park, *A generalization of a theorem of Janos and Edelstein*, Proc. Amer. Math. Soc. (to appear)

[7] ________, *Fixed points of f-contractive maps*, Rocky Mountain J. of Math. (to appear)

[8] C. S. Wong, *Generalized contractions and fixed point theorems*, Proc. Amer. Math. Soc. **42** (1974), 409-417. MR **48** #9692.

Seoul National University

and

The Korea Military Academy