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OPERATOR ALGEBRAS ON HILBERT SPACE

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1. Introduction.

Throughout the paper H denotes a nonzero complex Hilbert space and $B(H)$ the algebra of all bounded operators on H. A subalgebra B of $B(H)$ is called an algebra of finite strict multiplicity, if there exists a finite set $\{x_1, \cdots, x_n\} \subset H$ such that the linear span of $\{T_i x_i : T_i \in B, i=1, \cdots, n\}$ is exactly H. In this case, let us call $\{x_1, \cdots, x_n\}$ the multiplicity set for B.

In particular, when the multiplicity set for B is a set consisting of a single vector x, we say that B is a strictly cyclic operator algebra with a strictly cyclic vector x. If, in addition, $T \in B$ and $Tx=0$ imply that $T=0$, then we call that B is a strictly cyclic separated algebra. The aim of this paper is to generalize, in terms of algebras of finite strict multiplicity, various results obtained by B. Barnes [1], M. Embry ([3] [4] [5]), D. Herrero ([7]) and A. Lambert ([9] [10] [11]).

2. The structure of a selfadjoint algebra of finite strict multiplicity.

A structure theorm of a selfadjoint strict cyclic operator algebra was obtained by M. Embry ([5] p. 54 Theorem 3). In this section, we will extend her result to the setting of a reductive algebra of finite strict multiplicity (Theorem 1). Recall that an operator algebra on H is called reductive, if every closed invariant subspace is reducing. A selfadjoint algebra is always reductive. For $A \subset B(H)$, A' denotes the commutant of A in $B(H)$. An element T of a strongly closed subalgebra B of $B(H)$ is called a generator of B, if the smallest strongly closed subalgebra of $B(H)$ containing T and the identity operator I is exactly B. For $T \in B(H)$, $\sigma(T)$ will denote the spectrum of T relative to $B(H)$, and $\rho(T)$ the resolvent set of T. If the smallest strongly closed subalgebra of $B(H)$ containing T, I and $\{(T-\lambda)^{-1} : \lambda \in \rho(T)\}$ is B, we call T a rational generator of B. Hence a generator is always a rational generator.

If B is a subalgebra of $B(H)$ and n a positive integer, let $M_n(B)$ be the collection of all $n \times n$ matrices with entries in B, which can be regarded as

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a subalgebra of $B(H^{(n)})$, where $H^{(n)}=H\oplus\cdots\oplus H$ (n copies). Then the following facts are easy to check ([11] p. 960).
(1) If we put $B^{(n)}=\{T^{(n)}=T\oplus\cdots\oplus T : T\in B\}$, then $(M_n(B))'=(B')^{(n)}$.
(2) If A is an algebra of finite strict muliplicity with a multiplicity set $\{x_1, \cdots, x_n\}$, then $M_n(A)$ is a strictly cyclic algebra on $H^{(n)}$ with the strictly cyclic vector $x_1\oplus\cdots\oplus x_n$.
We prove only (v) of the next lemma, since the rest are found in the proof of Lemma 3.1 ([11], p. 960).

LEMMA 1. *Let A be a norm closed algebra of finite strict multiplicity on H, let B be a strongly closed subalgebra of A'. Let $\{x_1, \cdots, x_n\}$ be a multiplicity set for A, and write $u_0=x_1\oplus\cdots\oplus x_n$. Then,*

(i) *$K=B^{(n)}u_0$ is a closed subspace of $H^{(n)}$.*

(ii) *$B^{(n)}|K$ is a strictly cyclic separated algebra.*

(iii) *The mapping $T\to T^{(n)}|K$ is a continuous algebra isomorphism on B onto $B^{(n)}|K$, with respect to the norms.*

(iv) *Moreover, if T is a generator (or rational generator) of B, then $T^{(n)}|K$ is a generator (or rational generator, respectively) of $B^{(n)}|K$.*

(v) *The mapping $\theta : T\to Tx_1\oplus\cdots\oplus Tx_n$ is a bicontinuous linear isomorphism on B onto K, with respect to the norms. In fact, there exists a constant $k<\infty$ such that*

$$\|T\|\leq k\|Tx_1\oplus\cdots\oplus Tx_n\|, \text{ for all } T\in B.$$

Proof. (v) First the continuity of θ is clear. As we remarked above, $M_n(A)$ is a strictly cyclic operator algebra on $H^{(n)}$ with the strictly cyclic vector $u_0=x_1\oplus\cdots\oplus x_n$, and $(M_n(A))'=(A')^{(n)}$. By a result of Embry ([3] p. 444 Lemma 2.1, [11] p. 960 (3)), there exists a positive constant $k<\infty$ such that $\|T\|=\|T^{(n)}\|\leqslant k\|T^{(n)}u_0\|$, for every $T^{(n)}\in(A')^{(n)}=(M_n(A))'$ (i.e., for every $T\in A'$).
Thus $\theta : T\to Tx_1\oplus\cdots\oplus Tx_n$ is a norm continuous surjection. *If* $T^{(n)}u_0=0$, then $T=0$. Hence θ is one to one as well. The proof is completed by the open mapping theorem. Q. E. D.

The next lemma extends Theorem 2 of Embry ([4], p. 46). The key of the proof comes from a result of Herrero ([6], p. 78 Lemma 1) which asserts that a dense linear invariant manifold of an algebra of finite strict multiplicity is the whole space H. The proof of Lemma 2 can be easily done by using Zorn's lemma, as in the Embry's original proof in [4]. In this paper a subspace of H means always a closed one.

LEMMA 2. *Let A be a subalgebra of $B(H)$ of finite strict multiplicity. For any given invariant subspace N of A such that $N\neq H$, there exists a maximal invariant subspace M of A such that $N\subset M\neq H$.*

REMARK 1. The above lemma remains valid, if the term "invariant" is replaced by "hyper-invariant".

COROLLARY 1. *Let A be a norm closed subalgebra of $B(H)$ of finite strict multiplicity. For any nonzero reducing subspace N of A, there exists a nonzero minimal reducing subspace M of A contained in N.*

The following lemma extends Lemma 1 of Embry ([5], p. 54).

LEMMA 3. *Let A be a norm closed algebra of finite strict multiplicity on H. Then each collection $\{E_j\}$ of mutually orthogonal nonzero projections in A' is a finite collection.*

Proof. We apply (v) of Lemma 1 and proceed as in the proof of Embry ([5] p. 54).

COROLLARY 2. *Every normal element $T \varepsilon A'$ has the finite spectrum.*

Proof. It is similar to the proof of Corollary 2 of ([5] p. 54).

We are now ready to state the desired extension of Theorem 3 ([5] p. 54).

THEOREM 1. *Let A be a reductive algebra of finite strict multiplicity on a Hilbert space H. Then there existe a finite orthogonal decomposition $\{M_k\}^n_{k=1}$ of H with nonzero subspaces M_k such that each M_k reduces A and each $A|M_k$ is an algebra which has no proper invariant subspace.*

Proof. We combine Corollary 1 and Lemma 3 to conclude that there exists a finite orthogonal decomposition $\{M_k\}^n_{k=1}$ of H such that M_k reduces A and $A|M_k$ no longer has a proper invariant subspace. Q. E. D.

Analogies to Theorem 4 and Corollary 5 of [5] can be easily obtained, but omitted here.

3. Multiplicative linear functionals.

First we consider the norm dual space of a strongly closed subalgebra B of $B(H)$ such that $B \subset A'$, where A is a norm closed algebra of finite strict multiplicity. The next lemma is regarded as an extension of Lemma 3. 3 in ([10] p. 719) to the direction of multiplicity set, dropping the hypothesis that B is abelian.

LEMMA 4. *Let A be a norm closed subalgebra of finite strict multiplicity on H with the multiplicity set $\{x_1, \cdots, x_n\}$. Let B be a closed subalgebra of A' with respect to the strong operator topology. Then, f is a bounded linear functional on B if and only if f is of the form*

$$f(T)=(Tx_1, Sx_1)+\cdots+(Tx_n, Sx_n), \quad \textit{for some } S\in B.$$

The operator $S\in B$ is uniquely determined by f and the mapping $f\to S$ is a bicontinuous conjugate linear isomorphism on the norm dual space of B onto the normed space B.

Proof. Let θ be as in (v) of Lemma 1. Then, for each $T\in B$, $f(T)=f(\theta^{-1}(Tx_1\oplus\cdots\oplus Tx_n))$. Since $f\circ\theta^{-1}$ is a bounded linear functional on the Hilbert space K, we see that there exists $S\in B$ such that

$$\begin{aligned} f(T)&=(Tx_1\oplus\cdots\oplus Tx_n, Sx_1\oplus\cdots\oplus Sx_n)\\ &=(Tx_1, Sx_1)+\cdots+(Tx_n, Sx_n). \end{aligned}$$

An easy computation shows that

$|f(T)\leqslant(\|x_1\oplus\cdots\oplus x_n\|^2\|S\|)\|T\|$, that is, $\|f\|\leqslant\|x_1\oplus\cdots\oplus x_n\|^2\|S\|$. If $S_1\in B$ is another such S, then one can check immediately that

$(S_1-S)x_1=0, \cdots, (S_1-S)x_n=0$. Hence $(S_1-S)(A_1x_1+\cdots+A_nx_n)=0$, for all $A_1, \cdots, A_n\in A$, showing $S_1=S$. The conjugate linearity of $f\to S$ is easy to check. The rest of the proof is carried out by the open mapping theorem.

COROLLARY 3. *The norm dual space of B is bicontinuously isomorphic onto the complex vector space $B^*=\{T^* : T\in B\}$ with respect to the norms.*

When A is an abelian strictly cyclic operator algebra, the necessity implication of the following lemma was obtained by Lambert ([10] pp. 718-720). We borrow his idea to get into a more general situation.

LEMMA 5. *Let A be a norm closed strictly cyclic operator algebra on H and let B be a strongly closed subalgebra of A'. Let λ be a complex number. Then there exists a nonzero multiplicative (bounded) linear functional on B such that $f(T)=\lambda$ for some $T\in B$ if and only if λ is an eignvalue for a common eigenvector for all $T^*\in B^*$. More precisely, the last statement means that there exists a nonzero function $f: B\to C$ and a nonzero vector $z\in H$ such that*

$$T^*z=\overline{f(T)}z, \quad \textit{for all } T\in B, \textit{ and}$$

$f(T_b)=\lambda$ for some $T_b\in B$.

Proof. Let x_1 be a strictly cyclic vector for A. We apply (v) of Lemma 1 and Lemma 4 with $n=1$ to the present case. Then we can proceed as follows. Let f be a nonzero multiplicative linear functional on B such that $f(T_b)=\lambda$, for some $b\in H$. Here we put $T_y=\theta^{-1}(y)$, for $y\in H=K$. Then there exists a nonzero vector $z\in H$ such that

$$f(T_y)=(y, z), \quad \text{for all } y\in H,$$

so that $f(T_b)=(b, z)=\lambda$.

For $y, t\in H$, let $x=T_yt$. By noticing $T_tx_1=t$, for every $t\in H$, we see that

$T_x x_1 = T_y T_t x_1$. Thus $T_x = T_y T_t$. Then $f(T_x) = f(T_y) f(T_t)$ i.e., $(x, z) = (y, z)(t, z)$, for all $y, z \in H$. This shows that

$$(T_y t, z) = ((y, z)t, z) \text{ for all } t \in H,$$

$(t, T_y^* z) = (t, (z, y)z)$, for all $t \in H$, showing $T_y^* z = (z, y)z$, for all $y \in H$.

Since $z \neq 0$, (z, y) is an eigenvalue of T_y^* for all $y \in H$, and z is a common eigenvector of B^*. Then f is the required function, by noticing $f(T_y) = (y, z)$ and $f(T_b) = (b, z) = \lambda$.

Conversely let $T_y^* z = f_1(y) z$, for all $y \in H$ and $f_1(b) = \lambda$, for some $b \in H$, where z is a fixed vector of $\|z\| = 1$ and $f_1 : H \to C$ is a nonzero function. We define $f(T_y) = f_1(y)$, for all $T_y \in B$. Then it is easy to check that f is a well defined bounded multiplicative linear functional on B such that $f(T_b) = \lambda$. Q. E. D.

We are now ready to state the extension of the above lemma as follows.

THEOREM 2. *Let A be a norm closed algebra of finite strict multiplicity set $\{x_1, \cdots, x_n\}$. Let B be a strongly closed subalgebra of A'. Let $\lambda \in \boldsymbol{C}$. Then the same assertion holds as in Lemma* 1.

Proof. As we remarked just before Lemma 1, $M_n(A)$ is a norm closed strictly cyclic algebra on $H^{(n)}$ with the strictly cyclic vector $u_0 = x_1 \oplus \cdots \oplus x_n$, and $(M_n(A))' = (A')^{(n)}$. Thus $B^{(n)} \subset (A')^{(n)} = (M_n(A))'$. A simple verification shows that $B^{(n)}$ is strongly closed in $B(H^{(n)})$.

Let $\phi : B \to B^{(n)}$ be the canonical map defined by $\phi(B) = B^{(n)}$. Then $f(T) = (f \circ \phi^{-1})\ (T^{(n)})$, for all $T \in B$. Thus f is a nonzero multiplicative bounded linear functional on B such that $f(T_0) = \lambda$ if and only if $f \circ \phi^{-1}$ is a nonzero multiplicative bounded linear functional on $B^{(n)}$ such that $(f \circ \phi^{-1})(T_0^{(n)}) = \lambda$.

Applying Lemma 5 to the fact $B^{(n)} \subset (M_n(A))'$, we see that the last statement holds if and only if $\bar{\lambda}$ is an eigenvalue for a common eigenvector for all $(T^*)^{(n)} \in (B^{(n)})^*$, since $(B^{(n)})^* = (B^*)^{(n)}$ as operator algebras on $H^{(n)}$. Then $(T^*)^{(n)} w = \overline{f(T)} w$, for some $w \neq 0$, $w \in H^{(n)}$, for all $T \in B$, and $f(T_0) = \lambda$.

If we write $w = z_1 \oplus \cdots \oplus z_n \in H^{(n)}$, then, for all $T \in B$, we have

$$T^* z_i = \overline{f(T)} z_i, \text{ for each } i = 1, 2, \cdots, n.$$

But some $z_i \neq 0$, and for this i, z_i is a common eigenvector for B^* and

$$T_0^* z_i = \bar{\lambda} z_i.$$

This proves one direction of the assertion. The converse part of the proof is similar to the corresponding part of the proof of Lemma 5 and omitted.

Q. E. D.

REMARK 2. In the proof of sufficiency of the above, suppose we let $z \in H$, $\|z\|=1$ and $T^*z=f(T)z$ for all $T \in B$. Then

(1) $f(T)=(Tz, z)$, for all $T \in B$. Conversely, let f be the mapping defined by (1) such that f is nonzero and multiplicative, and $f(T_0)=\lambda$, for some $T_0 \in B$. Then a close examination of the proof of necessity of Theorem 2 shows that such a z can always be chosen so that z is a common eigenvector of B^* and $\|z\|=1$.

4. Spectral properties

The next is an extension of Theorem 4 ([1] p. 482).

THEOREM 3. *Let $T \in B(H)$ be a rational generator of a strongly closed abelian algebra B of finite strict multiplicity. If $\lambda \in \sigma(T)$ and $|\lambda|=\|T\|$, then λ is an isolated point of $\sigma(T)$.*

Proof. We keep the notation as in Lemma 1, by putting $B=A$. Then by Lemma 1, $B^{(n)}|K$ is a rational generator of $B^{(n)}|K$.
Thus, $\sigma(T)$, the spectrum of T relative to $B(H)$ = the spectrum of $T^{(n)}|K$ relative to $B^{(n)}|K$ = the spectrum of $T^{(n)}|K$ relative to $B(K)$. It is immediate to see that $\|T^{(n)}|K\| \leqslant \|T\|$. By hypothesis, $\lambda \varepsilon$ the spectrum of $T^{(n)}|K$ relative to $B(K)$, so that $|\lambda|=\|T^{(n)}\|K\||$. By the same argument as in the proof of Theorem 4 ([1] p. 482), we see that λ is an isolated point of the spectrum of $T^{(n)}|K$ relative to $B(K)$. It follows that λ is also an isolated point of $\sigma(T)$. Q. E. D.

REMARK 3. As we shall see in the proof of the next Theorem 4, the complex $\lambda \in \sigma(T)$ in Theorem 3 is actually an eigenvalue of T.

An operator $T \in B(H)$ is called paranormal if $\|Tx\|^2 \leqq \|T^2x\|\|x\|$, for all $x \in H$. For such an operator T, the spectral radius $r(T)$ equals the norm $\|T\|$, and any restriction $T|M$ to an invariant subspace $M(\neq(0))$ of T is also paranormal. It can be easily shown that a hyponormal operator T(i. e., $TT^* \leqslant T^*T$) is paranormal, but not conversely in general. The next theorem extends Corollary 3. 4 ([11] p. 962) and Theorem 8 ([1] p. 484).

THEOREM 4. *Let $T \in B(H)$ be a rational generator of a strongly closed abelian algebra B of finite strict multiplicity, acting on an infinite dimensional Hilbert space H. Thern T can not be a paranormal openator.*

Proof. Since $r(T)=\|T\|$, we find $\lambda \in \sigma(T)$ such that $|\lambda|=\|T\|$. Then λ is an isolated point of $\sigma(T)$. By Theorem 2, $\bar{\lambda}$ is an eigenvalue of T^*. It follows that λ is a reducing eigenvalue of T, that is, there is $x \neq 0$ in H

such that $Tx=\lambda x$ and $T^*x=\bar{\lambda}x$([8] p. 233 Satz 2). Let H_1, H_2 be the closed invariant subspaces of T associated with $\sigma(T)\sim\{\lambda\}$ and $\{\lambda\}$ respectively which come about in the Riesz decomposition for T([12] p. 421, [2] p. 574). Then it is not hard to see that $B|H_1$, $B|H_2$ are also strongly closed algebras of finite strict multiplicity with rational generators $T|H_1$, $T|H_2$ respectively. Then clearly, $B|H_2=\{\lambda I_2 : \lambda\in \boldsymbol{C}$, where I_2 is the identity operator on $H_2\}$, We apply the same argument to $B|H_2$ instead of B, recalling that $T|H_2$ is a paranormal rational generator of $B|H_2$. Continuing this process, we can deduce that T is a normal operator whose spectrum consists of reducing eigenvalues, whose eigenvectors generate H. By Corollary 2, $\sigma(T)$ is a finite set. Then we clearly see that T can not be a rational generator of a strongly closed algebra of finite strict multiplicity. Q. E. D.

The next is a strong improvement of Corollary 3 ([1] p. 482) and Corollary 2 ([7] p. 15).

THEOREM 5. *Let $T\in B(H)$ be a rational generator of a strongly closed algebra B of finite strict multiplicity. Then, $\sigma(T)=\sigma(T^*)^*=\sigma_p(T^*)^*$, where $X^*=\{\bar{\lambda}\in \boldsymbol{C} : \lambda\in X\subset \boldsymbol{C}\}$ and $\sigma_p(\cdot)$ denotes the point spectrum.*

Proof. We put $A=B$ and apply Theorem 2. Recall that $\lambda\in\sigma(T)$ if and only if there exists a nonzero multiplicative linear functional f on B such that $f(B)=\lambda$ ([13] p. 265(e) of 11. 5 Theorem). By Theorem 2, we see that the last statement is equivalent to saying that $\lambda\in\sigma_p(T^*)$. Hence $\sigma(T)=\sigma_p(T^*)^*$. Now if $\mu\in\sigma(T^*)$, then $\bar{\mu}\in\sigma(T^*)^*=\sigma(T)$. Thus $\bar{\mu}\in\sigma_p(T^*)^*$, showing $\mu\in\sigma_p(T^*)$. Q. E. D.

After this paper was completed, Prof. Peter Rosenthal kindly informed that our Theorem 1 also follows from his joint work with A. Feintuch (Israel Journal of Math. (15) No. 2, (1973), Corollary 4 and Remark (i) p. 135).

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