J. Korean Math. Soc.
Vol.14, No.1, 1977.

NOTE ON INFINITESIMAL CONFORMAL VARIATIONS

BY U-HANG KI

§0. Introduction.

Infinitesimal variations of submanifolds of a Riemannian manifold have studied by Chen ([1]), Ki ([5]), Pak ([5]), Yano ([1], [2], [4], [5]) and many authors. In his paper [4] Yano found the necessary and sufficient condition for a variation to be isometric in a compact submanifold.

The purpose of the present note is to study the necessary and sufficient condition for an infinitesimal variation to be conformal.

§1. Structure equations of submanifolds.

Let M^m be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and denote by g_{ji}, Γ_{ji}^h, ∇_j, $K_{kji}{}^h$ and K_{ji} the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to Γ_{ji}^h, the curvature tensor and Ricci tensor of M^m respectively, where here and in the sequel the indices $h, j, i, k, \cdots$ run over the range $\{1, 2, \cdots, m\}$.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by g_{cb}, Γ_{cb}^a, ∇_c, $K_{dcb}{}^a$ and K_{cb} the corresponding quantities of M^n respectively, where here and in the sequel the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We suppose that M^n is isometrically immersed in M^m by the immersion i : $M^n \to M^m$ and identify $i(M^n)$ with M^n itself. We represent the immersion by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h$ ($\partial_b = \partial/\partial y^b$). Then $B_b{}^h$ are n linearly independent vectors of M^m tangent to M^n. Since the immersion i is isometric, we obtain

$$(1.1) \qquad g_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

We denote by $C_y{}^h$ $m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel the indices x, y, z run over the range $\{n+1, n+2, \cdots, m\}$. Then the metric tensor of the normal bundle of M^n is given by

$$(1.2) \qquad g_{zy} = C_z{}^j C_y{}^i g_{ji}.$$

Received by the editors Apr. 15, 1977.

It is well known that Γ_{cb}^a and Γ_{ji}^h are related by

$$\Gamma_{cb}^a=(\partial_c B_b^h+\Gamma_{ji}^h B_c^j B_b^i)B^a{}_h, \tag{1.3}$$

where $B^a{}_h=B_b^i g^{ba} g_{ih}$, $(g^{ba})=(g_{ba})^{-1}$, and the components Γ_{cy}^x of the connection induced in the normal bundle are given by

$$\Gamma_{cy}^x=(\partial_c C_y^h+\Gamma_{ji}^h B_c^j C_y^i)C^x{}_h, \tag{1.4}$$

where $C^x{}_h=C_y^i g^{yx} g_{ih}$, g^{yx} being contravariant components of the metric tensor g_{yx} of the normal bundle.

If we denote by $\nabla_c B_b^h$ and $\nabla_c C_y^h$ the van der Waerden-Bortolotti covariant derivatives of B_b^h and C_y^h along the M^n respectively, that is, if we put

$$\nabla_c B_b^h=\partial_c B_b^h+\Gamma_{ji}^h B_c^j B_b^i-\Gamma_{cb}^a B_a^h \tag{1.5}$$

and

$$\nabla_c C_y^h=\partial_c C_y^h+\Gamma_{ji}^h B_c^j C_y^i-\Gamma_{cy}^x C_x^h, \tag{1.6}$$

then we can write equations of Gauss and those of Weingarten in the form

$$\nabla_c B_b^h=h_{cb}^x C_x^h, \tag{1.7}$$

$$\nabla_c C_y^h=-h_c{}^a{}_y B_a^h, \tag{1.8}$$

respectively, where h_{cb}^x are the second fundamental tensor of M^n with respect to the normals C_x^h and $h_c{}^a{}_x=h_{cbx}g^{ba}$.

Equations of Gauss, Codazzi and Ricci are respectively

$$K_{dcb}{}^a=K_{kji}{}^h B_d^k B_c^j B_b^i B^a{}_h+h_d{}^a{}_x h_{cb}^x-h_c{}^a{}_x h_{db}^x, \tag{1.9}$$

$$0=K_{kji}{}^h B_d^k B_c^j B_b^i C^x{}_h-(\nabla_d h_{cb}^x-\nabla_c h_{db}^x), \tag{1.10}$$

and

$$K_{dcy}{}^x=K_{kji}{}^h B_d^k B_c^j C_y^i C^x{}_h+(h_{de}^x h_c{}^e{}_y-h_{ce}^x h_d{}^e{}_y), \tag{1.11}$$

where $K_{dcy}{}^x$ is the curvature tensor of the connection induced in the normal bundle.

§2. Variations on a submanifold.

We now consider a variation of M^n in M^m given by $\bar{x}^h=x^h+f^h(y)\varepsilon$, where $g_{ji}f^j f^i>0$ and ε is an infinitesimal. We then have

$$\bar{B}_b^h=B_b^h+(\partial_l f^\cdot)\varepsilon, \tag{2.1}$$

where $\bar{B}_b^h=\partial_b \bar{x}^h$ are n linearly independent vectors tangent to the deformed submanifold at the deformed point $(\bar{x}^h)$. If we displace $\bar{B}_b^h$ parallely from the point $(\bar{x}^h)$ to (x^h), we then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x+f\varepsilon) f_j B_b{}^i \varepsilon,$$

at the point (x^h), or

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b f^h)\varepsilon, \tag{2.2}$$

neglecting the terms of order higher than one with respect to ε, where

$$\nabla_b f^h = \partial_b f^h + \Gamma_{ji}{}^h B_b{}^j f^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{2.3}$$

we have from (2.2)

$$\delta B_b{}^h = (\nabla_b f^h)\varepsilon. \tag{2.4}$$

If we put

$$f^h = f^a B_a{}^h + f^x C_x{}^h, \tag{2.5}$$

we have

$$\nabla_b f^h = (\nabla_b f^a - h_b{}^a{}_x f^x) B_a{}^h + (\nabla_b f^x + h_{ba}{}^x f^a) C_x{}^h \tag{2.6}$$

because of (1.7) and (1.8).

From (2.3), (2.4) and (2.6), we obtain

$$\tilde{B}_b{}^h = [\delta_b^a + (\nabla_b f^a - h_b{}^a{}_x f^x)\varepsilon] B_a{}^h + (\nabla_b f^x + h_{ba}{}^x f^a) C_x{}^h \varepsilon. \tag{2.7}$$

Now applying the operator δ to (1.1) and using (2.4), (2.6) and $\delta g_{ji} = 0$, we have (cf. [4])

$$\delta g_{cb} = (\nabla_c f_b + \nabla_b f_c - 2h_{cbx} f^x)\varepsilon, \tag{2.8}$$

where $f_b = g_{ba} f^a$, from which

$$\delta g^{ba} = -(\nabla^b f^a + \nabla^a f^b - 2h^{ba}{}_x f^x)\varepsilon. \tag{2.9}$$

A variation of a submanifold for which $\delta g_{cb} = 0$ is said to be isometric and that for which δg_{cb} is proportional to g_{cb} is said to be conformal (cf. [4]).

Thus we have from (2.8)

In order for a variation of a submanifold to be isometric or conformal, it is necessary and sufficient that

$$\nabla_c f_b + \nabla_b f_c - 2h_{cbx} f^x = 0, \tag{2.10}$$

or

$$\nabla_c f_b + \nabla_b f_c - 2h_{cbx} f^x = 2\lambda g_{cb}, \tag{2.11}$$

respectively λ being a certain function.

We denote by $\bar{C}_y{}^h$ $m-n$ mutually orthogonal unit normals to the deformed submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $C_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h=\bar{C}_y{}^h+\Gamma_{ji}{}^h(x+f\varepsilon)f^jC_y{}^i\varepsilon.$$

We put

$$\delta C_y{}^h=\tilde{C}_y{}^h-C_y{}^h \tag{2.12}$$

and assume that $\delta C_y{}^h$ is of the form

$$\delta C_y{}^h=(f_y{}^aB_a{}^h+f_y{}^xC_x{}^h)\varepsilon. \tag{2.13}$$

Applying the operator δ to $B_c{}^jC_y{}^ig_{ji}=0$ and using (2.4), (2.6), (2.13) and $\delta g_{ji}=0$, we obtain

$$(\nabla_bf_y+h_{bay}f^a)+f_{yb}=0,$$

where $f_{yb}=f_y{}^cg_{cb}$, or

$$f_y{}^a=-(\nabla^af_y+h_b{}^a{}_yf^b). \tag{2.14}$$

Applying also the operator δ to $C_x{}^jC_y{}^ig_{ji}=\delta_{xy}$ and using (2.13) and $\delta g_{ji}=0$, we have

$$f_{yx}+f_{xy}=0, \tag{2.15}$$

where $f_{yx}=f_y{}^zg_{zx}$.

We denote by $\bar{B}^a{}_i$ n covectors of the deformed submanifold corresponding to $B^a{}_i$ of the original submanifold and by $\tilde{B}^a{}_i$ the covectors obtained from $\bar{B}^a{}_i$ by parallel displacement of $\bar{B}^a{}_i$ from the point $(\bar{x}^h)$ to (x^h).

Then we have

$$\tilde{B}^a{}_i=\bar{B}^a{}_i-\Gamma_{ji}{}^h(x+f\varepsilon)f^j\bar{B}^a{}_h\varepsilon. \tag{2.16}$$

We put

$$\delta B^a{}_i=\tilde{B}^a{}_i-B^a{}_i. \tag{2.17}$$

Then applying the operator δ to $B_b{}^iB^a{}_i=\delta_b{}^a$, $C_y{}^iB^a{}_i=0$ and using (2.4) and (2.13), we find

$$\delta B^a{}_i=-(\nabla_bf^a-h_b{}^a{}_xf^x)B^b{}_i\varepsilon+(\nabla^af_x+h_b{}^a{}_xf^b)C^x{}_i\varepsilon. \tag{2.18}$$

From (2.16), (2.17) and (2.18), we find

$$\bar{B}^a{}_i=B^a{}_i+\{\Gamma_{ji}{}^hf^jB^a{}_h-(\nabla_bf^a-h_b{}^a{}_xf^x)B^b{}_i+(\nabla^af_x+h_b{}^a{}_xf^b)C^x{}_i\}\varepsilon. \tag{2.19}$$

We now put

(2.20) $$\bar{\Gamma}_{cb}{}^{a}=(\partial_c\bar{B}_b{}^h+\Gamma_{ji}{}^h(\bar{x})\bar{B}_c{}^j\bar{B}_b{}^i)\bar{B}^a{}_h$$

and

$$\delta\bar{\Gamma}_{cb}{}^{a}=\bar{\Gamma}_{cb}{}^{a}-\Gamma_{cb}{}^{a},$$

where $\bar{\Gamma}_{cb}{}^a$ are Christoffel symbols of the deformed submanifold. Then substituting (2.1) and (2.19) into (2.20), we obtain by a straightforward computation,

(2.21) $$\delta\Gamma_{cb}{}^{a}=\{(\nabla_c\nabla_b f^h+K_{kji}{}^h B_c{}^j B_b{}^i)B^a{}_h + h_{cb}{}^x(\nabla^a f_x+h_d{}^a{}_x f^d)\}\,\varepsilon,$$

from which, using equations (1.9) of Gauss and those (1.10) of Codazzi of the submanifold, we have (cf. [4])

(2.22) $$\delta\Gamma_{cb}{}^{a}=(\nabla_c\nabla_b f^a+K_{dcb}{}^a f^d)\,\varepsilon - \{\nabla_c(h_{bex}f^x)+\nabla_b(h_{cex}f^x)-\nabla_e(h_{cbx}f^x)\}\,g^{ea}\varepsilon.$$

A variation of a submanifold for which $\delta\Gamma_{cb}{}^a=0$ is said to be *affine* and that for which

(2.23) $$\delta\Gamma_{cb}{}^{a}=(\nabla_c\lambda)\delta_b^a+(\nabla_b\lambda)\delta_c^a-(\nabla^a\lambda)g_{cb}$$

is said to be *affine collinear*, where λ is a certain function.

We first prove

LEMMA 2.1. *In order for a variation of a submanifold to be affine collinear, it is necessary and sufficient that*

(2.24) $$\nabla_c(\delta g_{ba}-2\lambda g_{ba})=0.$$

Proof. From (2.8) and (2.24), we find

(2.25) $$\nabla_c\nabla_b f_a+\nabla_c\nabla_a f_b=2\nabla_c(h_{bax}f^x+\lambda g_{ba}),$$

from which, using the Ricci-identity

(2.26) $$\nabla_c\nabla_b f_a+\nabla_a\nabla_c f_b-K_{cabd}f^d=2\nabla_c(h_{bax}f^x+\lambda g_{ba}),$$

or substituting (2.25)

$$\nabla_c\nabla_b f_a-\nabla_a\nabla_b f_c-K_{cabd}f^d =2\nabla_c(h_{bax}f^x+\lambda g_{ba})-2\nabla_a(h_{bcx}f^x+\lambda g_{bc}).$$

If we take the skew-symmetric part of this with respect to a and b and make use of the Ricci-identity, then we obtain

$$\nabla_c\nabla_b f_a-\nabla_c\nabla_a f_b+K_{abcd}f^d-K_{cabd}f^d+K_{cbad}f^d =-2\nabla_a(h_{cbx}f^x+\lambda g_{cb})+2\nabla_b(h_{cax}f^x+\lambda g_{ca}),$$

from which, using (2.25) and first Bianchis identity,

$$\begin{aligned}(2.27)\qquad & \nabla_c\nabla_b f_a + K_{abcd}f^d \\ &= \nabla_c(h_{bax}f^x+\lambda g_{ba}) + \nabla_b(h_{cax}f^x+\lambda g_{ca}) - \nabla_a(h_{cbx}f^x+\lambda g_{cb}).\end{aligned}$$

Comparing (2.27) with (2.22), we have affine collinear. Reversing the argument, we conclude the converse assertion. Hence the lemma is proved.

§3. An integral formula.

THEOREM 3.1. *For a compact orientable submanifold M^n of a Riemannian manfiold, we haue the following integral formula:*

$$\begin{aligned}(3.1)\qquad \int_{M^n} [& \{\nabla^c\nabla_c f^a + K_c{}^a f^c - 2\nabla^c(h_c{}^a{}_x f^x) + \nabla^a(h_e{}^e{}_x f^x) + (n-2)\nabla^a\lambda\} f_a \\ & + (1/2)\{\nabla_c f_b + \nabla_b f_c - 2h_{cbx}f^x - 2\lambda g_{cb}\}\{\nabla^c f^b + \nabla^b f^c - 2h^{cb}{}_y f^y - 2\lambda g^{cb}\} \\ & + h_{cbx}f^x\{\nabla^c f^b + \nabla^b f^c - 2h^{cb}{}_y f^y - 2\lambda g^{cb}\} \\ & + 2\lambda\{\nabla_e f^e - h_e{}^e{}_x f^x - n\lambda\}]dV = 0\end{aligned}$$

for some function λ.

Proof. Since M^n is compact and orientable, we have the following integral formula:

$$\begin{aligned}(3.2)\qquad \int_{M^n} [& (\nabla^c\nabla_c f^a + K_c{}^a f^c) f_a + (1/2)(\nabla_c f_b + \nabla_b f_c)(\nabla^c f^b + \nabla^b f^c) \\ & - (\nabla_e f^e)^2]dV = 0,\end{aligned}$$

which is valid for any vector field f^a in M^n (cf. [4]). From (3.2), we find

$$\begin{aligned}(3.3)\qquad \int_{M^n} [& \{\nabla^c\nabla_c f^a + K_c{}^a f^c - 2\nabla^c(h_c{}^a{}_x f^x) + \nabla^a(h_e{}^e{}_x f^x) + (n-2)\nabla^a\lambda\} f_a \\ & + \{2\nabla^c(h_c{}^a{}_x f^x) - \nabla^a(h_e{}^e{}_x f^x) - (n-2)\nabla^a\lambda\} f_a \\ & + (1/2)(\nabla_c f_b + \nabla_b f_c)(\nabla^c f^b + \nabla^b f^c) - (\nabla_e f^e)^2]dV = 0,\end{aligned}$$

from which, by a straightforward computation, we obtain (3.1).

Now if a variation of the submanifold is conformal, we have (2.11) consequently

$$(3.4)\qquad (h^{cb}{}_y f^y)(\nabla_c f_b + \nabla_b f_c - 2h_{cbx}f^x) - 2\lambda g_{cb} = 0$$

and $$\nabla_e f^e - h_e{}^e{}_x f^x - n\lambda = 0,$$

or, equivalently

$$(3.5)\qquad (h^{cb}{}_y f^y)(\delta g_{cb}-2\lambda g_{cb})=0 \quad\text{and}\quad g^{cb}(\delta g_{cb}-2\lambda g_{cb})=0.$$

From Lemma 2.1, we have (2.23).

Conversely if (2.23) and (3.5) are satisfied, we have from Lemma 2.1 and (2.27),

$$(3.6)\qquad \nabla^c\nabla_c f^a+K_c{}^a f^c-2\nabla^c(h_c{}^a{}_x f^x)+\nabla^a(h_e{}^e{}_x f^x)+(n-2)\nabla^a\lambda=0.$$

Thus we see from the integral formula (3.1) that

$$\nabla_c f_b+\nabla_b f_c-2h_{cbx}f^x-2\lambda g_{cb}=0$$

and consequently the variation is conformal. Hence we have

THEOREM 3.2. *A necessary and sufficient condition for a variation of a compact orientable submanifold to be conformal is that the variation is affine collinear and* $(h^{cb}{}_y f^y)(\delta g_{cb}-2\lambda g_{cb})=0$, $g^{cb}(\delta g_{cb}-2\lambda g_{cb})=0$ *are satisfied.*

Now, if a variation of the submanifold to be isometric, we have (2.10) and conseqently (2.11) and (3.1) with $\lambda=0$.

Since an isometric variation is affine, we also have (2.22), form which

$$(3.7)\qquad \nabla^c\nabla_c f^a+K_c{}^a f^c-2\nabla^c(h_c{}^a{}_x f^x)+\nabla^a(h_e{}^e{}_x f^x)=0.$$

From (3.1) with $\lambda=0$ and (3.7) we have by the proof as in Theorem 3.2

THEOREM 3.3. ([4]) *In order for a variation of a compact orientable submanifold to be ismetric, it is necessary and sufficient that we have* (3.7) *and*

$$(h^{cb}{}_y f^y)\delta g_{cb}=0, \qquad g^{cb}\delta g_{cb}=0.$$

References

[1] B.Y. Chen and K. Yano, *On the theory of normal variations*, to appear in J. Differential Geometry.

[2] K. Yano, *Sur la théorie des déformations infinitésimales*, J. of the Fac. of Sci. Univ. of Tokyo, **6**(1949), 1-75.

[3] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland Publ. Co., Amsterdam, (1957).

[4] K. Yano, *Infinitesimal variations of submanifolds*, to appear in Kōdai Math. Sem. Rep.

[5] K. Yano, U-Hang Ki and Jin Suk Pak, *Infinitesimal variations of the Ricci tensor*, to appear in Kōdai Math. Sem. Rep.

Kyungpook University and Tokyo Institute of Technology