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SEQUENCES OF QUASI-CONTRACTIONS AND FIXED POINTS

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In this paper we are concerned with the relationship between the convergence of certain maps and the convergence of their fixed points. Results in such direction are obtained by a number of authors [1], [2], [6], [7], [8], [9]. Our main results are extended versions of those of J. Achari [1] and of R. N. Mukherjee [6]. Some results of F. F. Bonsall [2], N. Muresan [7], S. B. Nadler [8], and S. Reich [9] are simultaneously extended.

We need some preliminary facts.

A selfmap g of a metric space (X, d) is called a *quasi-contraction* if there exists an $\alpha \in [0, 1)$ such that

(i) $d(gx, gy) \leqslant \alpha \max\{d(x, y), \; d(x, gx), \; d(y, gy), \; d(x, gy), \; d(y, gx)\}$

for all $x, y \in X$. A selfmap g of X is called a *generalized contraction* if there exist $\beta_i \in [0, 1)$, $i=1, 2, 3, 4, 5$, $\sum_{i=1}^{5} \beta_i < 1$, such that

(ii) $d(gx, gy) \leqslant \beta_1 d(x, y) + \beta_2 d(x, gx) + \beta_3 d(y, gy) + \beta_4 d(x, gy) + \beta_5 d(y, gx)$

for all $x, y \in X$. Clearly a generalized contraction is always a quasi-contraction.

S. Massa [5] and Lj. B. Ciric [3] showed that a quasi-contraction g of a complete metric space has a unique fixed point u and $\lim_{n\to\infty} g^n x = u$ for all $x \in X$.

A pair (g, h) of selfmaps of (X, d) is called a *quasi-contraction* if there exists an $\alpha \in [0, 1)$ such that

(iii) $d(gx, hy) \leqslant \alpha \max\{d(x, y), \; d(x, gx), \; d(y, hy), \; [d(x, hy) + d(y, gx)]/2\}$

for all $x, y \in X$. A pair (g, h) is called a *generalized contraction* if there exist $\beta_i \in [0, 1)$, $i=1, 2, 3, 4$, $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$ such that

(iv) $d(gx, hy) \leqslant \beta_1 d(x, y) + \beta_2 d(x, gx) + \beta_3 (y, hy) + \beta_4 [d(x, hy) + d(y, gx)]$

for all $x, y \in X$. Clearly a generalized contraction pair is always a quasi-contraction.

Ciric [4] showed that if (g, h) is a quasi-contraction of a complete met-

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ric space X then g and h have a unique common fixed point u and $\lim_{n\to\infty} x_n = u$ for all $x_0 \in X$, where $\{x_n\}$ is defined recursively by $x_1 = gx_0$, $x_2 = hx_1$, $x_3 = gx_2$, $x_4 = hx_3, \cdots$.

In (i) and (iii), α will be called the *control* constant.

Now we state our results.

THEOREM 1. *Let g_n be a selfmap of a metric space (X, d) with a fixed point u_n for each $n = 1, 2, 3, \cdots$ and g be a quasi-contraction of X with a fixed point u. If $\{g_n\}$ converges uniformly to g, then $\{u_n\}$ converges to u.*

Proof. Since g_n converges uniformly to g, for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$d(u_n, gu_n) = d(g_n u_n, gu_n) < \varepsilon(1-\alpha)/(1+\alpha)$$

for $n \geqslant N$, where α is the control constant for g. Then

$$\begin{aligned} d(u_n, u) = d(u_n, gu) &\leqslant d(u_n, gu_n) + d(gu_n, gu) \\ &\leqslant d(u_n, gu_n) + \alpha \max\{d(u_n, u), d(u_n, gu_n), d(u, gu), \\ &\qquad d(u_n, gu), d(u, gu_n)\} \\ &\leqslant d(u_n, gu_n) + \alpha \max\{d(u_n, u), d(u_n, gu_n), d(u, gu_n)\} \\ &\leqslant d(u_n, gu_n) + \alpha[d(u_n, u) + d(u_n, gu_n)] \end{aligned}$$

and hence we have

$$d(u_n, u) \leqslant [(1+\alpha)/(1-\alpha)] d(u_n, gu_n) < \varepsilon$$

for $n \geqslant N$.

Note that a fixed point of a quasi-contraction is always unique.

THEOREM 2. *Let d_n be a metric on a set X for each $n = 0, 1, 2, 3, \cdots$ and $\{d_n\}_{n=1}^{\infty}$ converge uniformly to $d = d_0$. Let g_n be a quasi-contraction of (X, d_n) with the control constant α_n for each $n > 0$. If $g : (X, d) \to (X, d)$ is the d-pointwise limit of $\{g_n\}_{n=1}^{\infty}$ and if $\alpha_n \to \alpha < 1$, then g is a quasi-contraction with the control constant α. Furthermore, if each g_n has a fixed point u_n and g has a fixed point u, then $\{u_n\}_{n=1}^{\infty}$ d-converges to u.*

Proof. For any $x, y \in X$, we have

$$d(gx, gy) \leqslant d(gx, g_n x) + d(g_n x, g_n y) + d(g_n y, gy).$$

Since d_n converges uniformly to d and $\alpha_n \to \alpha$, for any $\varepsilon > 0$, there exists $N > 0$ such that for $n \geqslant N$ we have $\alpha_n \leqslant \alpha + \varepsilon$ and $|d_n(x, y) - d(x, y)| \leqslant \varepsilon$ for all $x, y \in X$. Then

$$\begin{aligned} d(gx, gy) &\leqslant d(gx, g_n x) + d_n(g_n x, g_n y) + \varepsilon + d(g_n y, gy) \\ &\leqslant d(gx, g_n x) + \alpha_n \max\{d_n(x, y),\ d_n(x, g_n x),\ d_n(y, g_n y), \end{aligned}$$

$$d_n(x, g_n y), d_n(y, g_n x)\} + \varepsilon + d(g_n y, gy)$$
$$\leqslant d(gx, g_n x) + (\alpha+\varepsilon)\max\{d(x,y)+\varepsilon,\ d(x, g_n x)+\varepsilon,$$
$$d(y, g_n y)+\varepsilon, d(x, g_n y)+\varepsilon, d(y, g_n x)+\varepsilon\} + \varepsilon + d(g_n y, gy)$$

for all $x, y \in X$. Since $g_n x \xrightarrow[d]{} gx$ for all $x \in X$, we have

$$d(gx, gy) \leqslant \alpha \max\{d(x,y), d(x,gx), d(y,gy), d(x,gy), d(y,gx)\}$$

This shows that g is a quasi-contraction.

Furthermore, suppose g_n has a fixed point u_n for each $n>0$ and g has a fixed point u. Then for any ε, $0<\varepsilon<1-\alpha$, there exists $N>0$ such that for $n \geqslant N$ we have $\alpha_n \leqslant \alpha+\varepsilon$, $d(g_n u, gu) = d(g_n u, u) < \varepsilon$ and $|d_n(x,y) - d(x,y)| < \varepsilon$ for all $x, y \in X$. Therefore,

$$d(u_n, u) \leqslant d(g_n u_n, g_n u) + d(g_n u, u)$$
$$\leqslant d_n(g_n u_n, g_n u) + \varepsilon + \varepsilon$$
$$\leqslant \alpha_n \max\{d_n(u_n, u), d_n(u_n, g_n u_n), d_n(u, g_n u),$$
$$d_n(u_n, g_n u), d_n(u, g_n u_n)\} + 2\varepsilon$$
$$\leqslant \alpha_n [d_n(u_n, u) + d_n(u, g_n u)] + 2\varepsilon$$
$$\leqslant (\alpha+\varepsilon)[d(u_n, u) + d(u, g_n u) + 2\varepsilon] + 2\varepsilon$$
$$\leqslant (\alpha+\varepsilon) d(u_n, u) + 3\varepsilon(\alpha+\varepsilon) + 2\varepsilon$$

for $n \geqslant N$ and, hence,

$$d(u_n, u) \leqslant (3\varepsilon\alpha + 3\varepsilon^2 + 2\varepsilon)/(1-\alpha-\varepsilon).$$

Therefore, $u_n \xrightarrow[d]{} u$.

COROLLARY 3. *Let (X, d) be a complete metric space and g_n be a quasi-contraction with the control constant α_n for each $n=1, 2, \cdots$. Let u_n be the unique fixed point of g_n for each n. If $g: X \to X$ is the pointwise limit of $\{g_n\}_{n=1}^{\infty}$ and if $\alpha_n \to \alpha < 1$, then g is a quasi-contraction with the control constant α and* $\lim_{n\to\infty} u_n = u$, *the unique fixed point of g.*

Proof. Note that every quasi-contraction of a complete metric space has a unique fixed point. Then the proof follows from Theorem 2.

In [1] Achari obtained particular forms of Theorems 1, 2 and Corollary 3 for quasi-contractions g satisfying the condition

$$d(gx, gy) \leqslant \alpha \max\{d(x,y), [d(x,gx)+d(y,gy)]/2, [d(x,gy)+d(y,gx)]\}/2$$

instead of (i), and extended the results of Bonsall [2] and Nadler [8] for Banach contractions, that is, maps g satisfying (ii) with $\beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$.

In [7] Muresan obtained Theorem 1 for generalized contractions with $\beta_2 = \beta_3$. In [9] Reich obtained Corollary 3 for generalized contractions with $\beta_4 = \beta_5 = 0$.

Note also that we can extend Corollary 3 by assuming that X is g-orbitally complete [3].

Now we have our final result.

THEOREM 4. *Let (X, d) be a complete metric space and $\{g_n\}$, $\{h_n\}$ be two sequences of selfmaps of X such that (g_n, h_n) is a quasi-contraction with a control constant α_n for each $n=1, 2, \cdots$. If $g, h: X \to X$ are pointwise limit of $\{g_n\}$, $\{h_n\}$, respectively, and if $\alpha_n \to \alpha < 1$, then (g, h) is a quasi-contraction with α. Furthermore, the sequence of the unique common fixed point u_n of g_n and h_n converges to the unique common fixed point u of g and h.*

Proof. For any $x, y \in X$, we have

$$\begin{aligned} d(gx, hy) &\leqslant d(gx, g_n x) + d(g_n x, h_n y) + d(h_n y, hy) \\ &\leqslant d(g_n x, gx) + d(h_n y, hy) + \alpha_n \max\{d(x, y), d(x, g_n x), \\ &\quad d(y, h_n y), [d(x, h_n y) + d(y, g_n x)]\}/2. \end{aligned}$$

Since $g_n x \to gx$, $h_n y \to hy$ and $\alpha_n \to \alpha$ as $n \to \infty$, we have

$$d(gx, hy) \leqslant \alpha \max\{d(x, y), d(x, gx), d(y, hy), [d(x, hy) + d(y, gx)]/2\}.$$

Therefore, (g, h) is a quasi-contraction, and g and h have a unique common fixed point u since X is complete.

Let u_n be the unique common fixed point of g_n and h_n for each n. For each ε, $0 < \varepsilon < 1 - \alpha$, there exists $N > 0$ such that for $n \geqslant N$ we have $d(h_n u, u) = d(h_n u, hu) < (1 - \alpha - \varepsilon)\varepsilon / (1 + \alpha + \varepsilon)$ and $\alpha_n \leqslant \alpha + \varepsilon$. Then

$$\begin{aligned} d(u_n, u) &\leqslant d(g_n u_n, h_n u) + d(h_n u, u) \\ &\leqslant \alpha_n \max\{d(u_n, u), d(u_n, g_n u_n), d(u, h_n u), \\ &\qquad [d(u_n, h_n u) + d(u, g_n u_n)]/2\} + d(h_n u, u) \\ &\leqslant (\alpha + \varepsilon)[d(u_n, u) + d(u, h_n u)] + d(h_n u, u) \end{aligned}$$

and hence

$$d(u_n, u) \leqslant [(1 + \alpha + \varepsilon)/(1 - \alpha - \varepsilon)] d(h_n u, u) < \varepsilon.$$

This shows that $u_n \longrightarrow u$.

In [6], Mukherjee claimed Theorem 4 for a particular type of generalized contractions satisfying (iv) with $\beta_1 = \beta_4 = 0$, $\beta_2 = \beta_3$ for all $x, y \in X$, $x \neq y$.

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