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ON THE RADICAL OF A TOPOLOGICAL ABELIAN GROUP

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Introduction.

The radical of a topological abelian group was originally defined by F. B. Wright [2], then Kwang Chul Ha [1] gave an equivalent definition. Both authors used the idea of the radical mainly to obtain structure theorems for locally compact abelian groups. However, the examples given for the radical were only for the cases when the group was locally compact, compact, discrete or indiscrete.

In this Section 1 of this paper we investigate the radical in a general topological abelian group G and obtain many interesting results including restrictions as to which subgroups can be the radical of an abelian group. In Section 2 we consider abelian groups with a linear (i. e., "subgroup") topology. A list of some of these results follows:

a) If the torsion subgroup G_t of G is open, then G_t equals the radical of G.
b) The radical $T(G)$ of G is a pure subgroup of G.
c) $G/T(G)$ is torsion free.
d) If G has the Z-adic or p-adic topology, then $T(G)=G$.
e) For any topology the radical of Q or Z cannot be a proper subgroup.
f) If G has a subgroup topology, then $T(G)$ is its own radical.

We then show that if K is a subgroup of G and G/K is torsion free, then there is a topology on G such that $K=T(G)$. Many of these theorems are generalizations of the work in [1] and [2].

1. Notation, definitions and some examples.

Whenever the word group is used we shall mean abelian group. Let G be a topological abelian group and A any subset of G. If k is a positive integer, then we define the following sets: $kA=\{a_1+\cdots+a_k : a_i \in A\}$ and $-kA=k(-A)$ where $-A=\{-a: a\in A\}$. If W is an open set $O(W)=\bigcup_{k\in Z} kW$ where Z represents the set of integers (Z^+ shall be the positive integers). By G_t we shall mean the torsion subgroup of G. The definition of the radical of a topological abelian group given here is based on results obtained in [1] and

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[2].

DEFINITION 1: The radical $T(G)$ of a topological abelian group G is the set of all elements x such that for any neighborhood U_x of x, $O(U_x)$ contains 0. We now give some examples of radicals of topological abelian groups.

EXAMPLE 1 ([2], Theorem 4.1). Discrete groups. If G is a discrete abelian group, the radical $T(G)$ of G is the torsion subgroup G_t of G. Choosing $\{x\}$ as U_x, $x \in T(G)$ if and only if x has finite order.

EXAMPLE 2. Indiscrete groups. Obviously $T(G)=G$ in this case.
These two examples provide us with the following:

REMARK: The radical of a topological abelian group G lies between G and the torsion subgroup of G. To see this we observe that if G has two compatible group topologies J_1 and J_2 such that $J_1 \subset J_2$ and $T(J_1)$ and $T(J_2)$ represent the radicals of $G(J_1)$ and $G(J_2)$ respectively, then $T(J_2) \subset T(J_1)$. Our remark follows from the fact that the radical of a discrete group is the torsion subgroup and the radical of an indiscrete group is the group itself.

EXAMPLE 3. Compact groups. It is shown in [2] that if G is a compact group, then $T(G)=G$.

DEFINITION 2. Let G be a topological abelian group. If $G=T(G)$, G is said to be a radical group while if $T(G)=0$, G is said to be a radical free group. A subgroup is said to be a radical subgroup if it is a radical group in the relative topology.

DEFINITION 3. A subgroup H of an abelian group G is a pure subgroup of G if whenever $h=ng$ for $h \in H$, $n \in Z$ and $g \in G$, there is $h_1 \in H$ such that $h=nh_1$.

Theorems 1.1 through 1.5 which follow are some of the important results on radicals of general topological groups from [1] and [2]. The proofs can be found in [1].

THEOREM 1.1. *The radical $T(G)$ of a topological abelian group G is a closed subgroup of G.*

THEOREM 1.2. ([2], Theorem 4.7). *If G is a topological abelian group with radical $T(G)$, and if H is a closed subgroup of G such that $H \subset T(G)$, then the radical of G/H is $T(G)/H$. In particular, $G/T(G)$ is radical free.*

THEOREM 1.3 ([2], Theorem 4.8). *Let G be a topological abelian group with radical $T(G)$. If H is a closed subgroup of G such that G/H is radi-*

cal free, then $T(G)\subset H$.

THEOREM 1.4 ([4], Theorem 1.8). *Let G be a topological abelian group and H be a subgroup of G. If G/H and H are both radical groups, then so is G.*

THEOREM 1.5 ([1], Lemma 1.9). *Let G be a topological abelian group with radical* $T(G)$. *If* $T(G)$ *is open, then* $T(G)$ *is a radical subgroup of G.*

LEMMA 1.6. *Suppose H is a subgroup of G and H contains the torsion subgroup of G. Then H is a pure subgroup of G if and only if G/H is torsion free (i.e., no nonzero element has finite order).*

Proof: If G/H is torsion free, then clearly H is a pure subgroup of G. Now for the sufficiency suppose H is a pure subgroup of G and n is a nonzero integer such that $nx\in H$. Then there is $h\in H$ such that $nx=nh$ and $n(x-h)=0$. Hence $x-h\in G_t$. Since $G_t\leq H$ we have $x\in H$. Thus G/H is torsion free.

LEMMA 1.7. *Suppose G is a topological abelian group. If K is an open subgroup of G and G/K is torsion free, then* $T(G)\leq K$.

Proof: Since K is an open subgroup of G, K is closed and G/K is discrete. Since G/K is discrete and torsion free it is radical free. So K is a closed subgroup and G/K is radical free and by Theorem 1.3 $T(G)\leq K$.

THEOREM 1.8. *If G is a topological abelian group and K is a pure open subgroup of G such that* $G_t\leq K\leq T(G)$, *then* $K=T(G)$.

Proof: Since K is pure and contains G_t by Lemma 1.6 G/K is torsion free. By Lemma 1.7 $T(G)\leq K$, thus $K=T(G)$. From the above we have the following generalization of Theorem 4.1 of [2].

THEOREM 1.9. *If G is a topological abelian group and* G_t *is an open subgroup of G, then* $G_t=T(G)$.

Proof: The torsion subgroup G_t is open, G/G_t is torsion free and by Theorem 1.8 we have $G_t\leq T(G)$ so $G_t=T(G)$.

COROLLARY 1.10. *If G is a topological abelian group and* $T(G)$ *is open, the* $T(G)$ *is the smallest open subgroup containing* G_t.

LEMMA 1.11. *Suppose G is a topological abelian group with subgroups H and K such that* $H\leq K$. *If H is open, then K is open.*

Proof: K is the union of the cosets of H.

REMARK. In order that the radical is properly contained between G_t and

G we see that G_t cannot contain an open subgroup. The next theorem puts further restrictions on the subgroups of an abelian group which can be the radical for some topology.

THEOREM 1.12. *If G is a topological abelian group, then $T(G)$ is a pure subgroup of G.*

Proof: Suppose $nx \in T(G)$ for some $n \in Z$. Let U_x be a neighborhood of x, then $nx \in nU_x$. Since $nx \in T(G)$ and nU_x is a neighborhood of nx we have $0 \in 0(nU_x) = \cup_{k \in Z} k(nU_x) \subseteq \cup_{k \in Z} kU_x = 0(U_x)$. So $0 \in 0(U_x)$ and consequently $x \in T(G)$.

The following was proved in [1] in the case that G is locally compact. We see that it holds in the general case.

COROLLARY 1.13. *If G is a topological abelian group with radical $T(G)$, then $G/T(G)$ is torsion free.*

Let Q represent the additive group of the rational numbers.

COROLLARY 1.14. *The groups Q and Z are either radical groups or radical free groups for any group topology.*

Proof: Both Q and Z have the property that neither contains a proper pure subgroup.

2. The radical of groups which have a linear topology.

A great deal of study in abelian groups has been devoted to groups which have a linear topology, that is, groups which have a base of neighborhoods about 0 which consists of subgroups such that the cosets of these subgroups form a base of open sets. In considering the radical of such groups we discover that groups in some of the familiar linear topologies are radical groups. We then show that we can define a topology (not necessarily linear) for any non-torsion group so that any (proper) pure subgroup which contains the torsion subgroup is the radical. We begin with

LEMMA 2.1. *Let G be an abelian group with a linear topology and let H be an open subgroup of G, then $0(x+H) = \cup_{k \in Z}(kx+H)$ for $x \in G$.*

Proof: By definition $0(x+H) = \cup_{k \in Z} k(x+H)$ but $k(x+H) = \{kx+h : h \in H\}$. Thus $0(x+H) = \cup_{k \in Z}(kx+H)$.

THEOREM 2.2. *Suppose G is an abelian group with a linear topology and $T(G)$ is the radical of G, then $x \in T(G)$ if and only if for every open subgroup H of G there is $k \in Z^+$ such that $kx \in H$.*

Proof: If $x\in T(G)$. then for every open subgroup H of G, $x+H$ is a neighborhood of x and $0\in O(x+H)=U(nx+H)$. Hence $0=kx+h$ for some $k\in Z^+$ and $h\in H$ and thus $kx\in H$. Conversely, suppose $x\in G$ and for every open subgroup H of G there is $k\in Z^+$ such that $kx\in H$. Let U be a neighborhood of x, then there is an open subgroup H such that $x+H\subseteq U$ and hence $H=kx+H\subseteq O(x+H)\subseteq O(U)$. Thus $0\in O(U)$ which means $x\in T(G)$.

COROLLARY 2. 3. *Let G be an abelian group with a linear topology. Then $G=T(G)$ if and only if G/H is a torsion group for every open subgroup H.*

COROLLARY 2. 4. *If G is an abelian group with either the Z-adic or p-adic topology, then $G=T(G)$.*

Proof: A base for the neighborhoods of 0 in the Z-adic or p-adic topology is all subgroups $\{kx\mid x\in G \text{ and } k\in Z\}$ or $\{p^kx\mid x\in G \text{ and } k\in Z^+\}$ respectively.

So if we wish to find groups with a linear topology which are not radical groups, then these groups must contain pure open subgroups, that is, open subgroups which have torsion free quotient groups. We now show that there do exist topological abelian groups such that the radical is properly contained between the group and the torsion subgroup.

THEOREM 2. 5. *If G is an abelian group and K is a subgroup of G such that G/K is torsion free, then there is a topology on G such that $K=T(G)$.*

Proof: Let K have a topology such that K is its own radical. A topology for G is obtained by taking an open basis for 0 in K and using this same collection of sets as an open basis for 0 in G. In this topology on G, K is open and $T(K)\leq T(G)$. By our construction $K=T(K)$. So $K\leq T(G)$ and G/K is torsion free and by Lemma 1. 7 $T(G)\leq K$. Hence $K=T(G)$.

COROLLARY 2. 6. *If G is an abelian group and $G_t\leq K\leq G$ and K is pure subgroup of G, then there is a topology on G such that $K=T(G)$.*

Proof: By Lemma 1. 6 the quotient group G/K is torsion free.

REMARK. If the topology of K above is (is not) a linear topology, then the topology of G induced from K is (is not) a linear topology.

THEOREM 2. 7. *Suppose G is an abelian group and $G_t\leq K\leq G$. Then K is a pure subgroup of G if and only if there is a topology on G such that $K=T(G)$.*

Proof: The sufficiency follows from Corollary 2. 6 and the necessity from Theorem 1. 11.

3. Is $T(G)$ a radical group?

In [2] the following occurs: "The question naturally arises: is the radical itself a radical subgroup? In general this is an open question. For locally compact groups the answer is affirmative." Theorem 1.5 also provides an affirmative answer in the case when the radical is an open subgroup. Our next theorem shows that another sufficient condition is that the group have a linear topology.

THEOREM 3.1. *If G is an abelian group with a linear topology, then $T(G)$ is a radical subgroup.*

Proof: We will apply Theorem 2.2 to $T(G)$. Let K be an open subgroup of $T(G)$, then there is an open subgroup H of G such that $K=T(G)\cap H$. Let $x\in T(G)$, then by Theorem 2.2 there is $k\in Z^+$ such that $kx\in H$. Since $kx\in T(G)$ we have $kx\in T(G)\cap H=K$. Hence $T(G)$ is its own radical.

References

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