J. Korean Math. Soc.
Vol. 14, No. 1, 1977.

EXTREME POINTS OF CONVEX SETS OF BICONTRACTIONS ON l_∞

BY C. W. KIM AND J. I. UM

1. Introduction.

A linear operator A from l_∞ into itself with $\|A\|_1 \leqq 1$ and $\|A\|_\infty \leqq 1$ will be called a bicontraction on l_∞. Let $\mathcal{S}'$ denote the convex set of bicontractions on l_∞. Let $\mathcal{D}'$ denote the convex set of positive bicontractions A on l_∞ (called doubly substochastic operators or matrices), that is, $A \in \mathcal{S}'$ and $Ax \geqq 0$ for each x $(0 \leqq x \in l_\infty)$. Mauldon [5] gave a direct proof to a result of Kendall and Kiefer [2] that the set of extreme points of infinite doubly stochastic matrices is the set of infinite permutation matrices. The purpose of this paper is to give a direct proof to another result of Kendall and Kiefer [2] on the set of extreme points of $\mathcal{D}$ and a characterization of the set of extreme points of $\mathcal{S}'$.

We assume that $1 \leqq p \leqq \infty$. Let $[l_p]$ dence the vector space of bounded linear operators from l_p into itself. Note that $\mathcal{S}' \subset [l_1] \cap [l_\infty]$. Let $e_i = (\delta_{ij} : j = 1, 2, \cdots)$, where δ_{ij} is the Kronecker delta. Let $\langle f, g \rangle = \sum_i f(i) g(i)$ $(f \in l_1, g \in l_\infty)$.

Let $\mathfrak{X}$ denote the vector space of infinite real matrices (a_{ij}) such that $\sup_i \sum_j |a_{ij}| < \infty$ and $\sup_j \sum_i |a_{ij}| < \infty$. Then there exists a bijection between $[l_1] \cap [l_\infty]$ and $\mathfrak{X}$. For each $A \in [l_1] \cap [l_\infty]$, if we define the matrix (a_{ij}) by $a_{ij} = Ae_j(i)$ $(= \langle e_i, Ae_j \rangle)$, where $i, j = 1, 2, \cdots$, then

$$Ax(i) = \sum_j a_{ij} x(j) \quad (x \in l_\infty,\ i = 1, 2, \cdots), \tag{1}$$

$$\|A\|_1 = \sup_j \sum_i |a_{ij}| < \infty, \quad \|A\|_\infty = \sup_i \sum_j |a_{ij}| < \infty. \tag{2}$$

Conversely, each matrix (a_{ij}) in $\mathfrak{X}$ defines a unique operator A in $[l_1] \cap [l_\infty]$ satisfying (1) and (2). Thus, we shall identify $[l_1] \cap [l_\infty]$ with $\mathfrak{X}$ and, in particular, also write

$$\mathcal{S}' = \{(a_{ij}) \in \mathfrak{X} : \sup_j \sum_i |a_{ij}| \leqq 1,\ \sup_i \sum_j |a_{ij}| \leqq 1\}.$$

By the Riesz convexity theorem, we have that

$[l_1] \cap [l_\infty] \subset \bigcap_{1 \leq p \leq \infty} [l_p]$, so that $\mathcal{S}'$ may be topologized by the weak (strong) operator topology for $[l_p]$. The weak operator topology for $[l_p]$ will be denoted by the l_p-w. o. t. and the strong operator topology for $[l_p]$

Received by the editors Apr. 8, 1977.
This work was supported by NRC Grant A-4844.

by the l_p-s. o. t. Each $A=(a_{ij})$ in $[l_1]\cap[l_\infty]$ as an element of $[l_1]$ determines the adjoint A^* in $[l_\infty]$ which is represented by the transpose (a^*_{ij}) of the matrix (a_{ij}) as

$$A^*x(j)=\Sigma_i a^*_{ij}x(i)=\sum_i a_{ij}x(i) \quad (x\in l_\infty, j=1,2,\cdots)$$

with $\|A^*\|_\infty=\|A\|_1$ and $\|A^*\|_1=\|A\|_\infty$. It is easily seen that both $[l_1]\cap[l_\infty]$ and $\mathcal{S}'$ are self-adjoint, $[l_1]\cap[l_\infty]=([l_1]\cap[l_\infty])^*$ and $\mathcal{S}'=\mathcal{S}'^*$. By the l_1-strong* operator topology (the l_1-s*. o. t.) for $[l_1]\cap[l_\infty]$, we shall mean the topology induced by ε-neighbourhoods, an ε-neighbourhood of $A=(a_{ij})$ in $[l_1]\cap[l_\infty]$ as the set

$$\{B : \|(A-B)x_i\|_1<\varepsilon,\ \|(A^*-B^*)y_i\|_1<\varepsilon, i=1,2,\cdots,n\},$$

where $B=(b_{ij})\in[l_1]\cap[l_\infty]$; $x_1,\cdots,x_n, y_1,\cdots,y_n\in l_1$, or equivalently

$$\{(b_{ij}) : \textstyle\sum_k|a_{kj}-b_{kj}|<\varepsilon,\ \sum_k|a_{ik}-b_{ik}|<\varepsilon,\ (i,j=1,2,\cdots,n)\}.$$

For each doubly substochastic (d. s. s.) matrix $A=(a_{ij})$, $A\in\mathcal{D}'$, we see that $0\leq a_{ij}\leq 1$ $(i,j=1,2,\cdots)$, $\sum_k a_{kj}\leqq 1$ for each j, and $\sum_k a_{ik}\geqq 1$ for each i. A d. s. s. matrix (a_{ij}) is called weakly doubly stochastic (w. d. s.) or weak* doubly stochastic (w*. d. s.) according as $\sum_k a_{ik}=1$ for each i or $\sum_k a_{kj}=1$ for each j. Let $\mathcal{D}_w$ and $\mathcal{D}_w{}^*$ denote the convex set of w. d. s. matrices and the convex set of w*. d. s. matrices. A d. s. s. matrix (a_{ij}) such that $\sum_k a_{ik}=1$ for each i and $\sum_k a_{kj}=1$ for each j is called doubly stochastic (d. s.). If we denote by $\mathcal{D}$ the convex set of d. s. matrices, then $\mathcal{D}=\mathcal{D}_w\cap\mathcal{D}_w{}^*$. Define the sets $\mathcal{P}', \mathcal{P}_w$, and $\mathcal{P}_w{}^*$ as follows:

$$\mathcal{P}'=\{(a_{ij})\in\mathcal{D}' : a_{ij}=0 \text{ or } 1\ (i,j=1,2,\cdots)\},$$

$$\mathcal{P}_w=\mathcal{P}'\cap\mathcal{D}_w,\quad \mathcal{P}_w{}^*=\mathcal{P}'\cap\mathcal{D}_w{}^*.$$

Note that each matrix in $\mathcal{P}'$ has at most one entry 1 in each row and in each column with remaining entries equal to 0, and that each matrix in $\mathcal{P}_w$ $(\mathcal{P}_w{}^*)$ has precisely one entry 1 in each row (column) and at most one entry 1 in each column (row) with remaining entries equal to 0. Denote by $\mathcal{P}$ the set of infinite permutation matrices.

For each $A=(a_{ij})$ in $[l_1]$, the positive operator $|A| : l_1\to l_1$ defined by

$$|A|x(i)=\sup\{|Ay(i)| : |y|\leqq x, y\in l_1\},$$

where $0\leqq x\in l_1, i=1,2,\cdots,$
is called the (linear) modulus of A. It follows readily that $|A|=(|a_{ij}|)$, $|Ax(i)|\leq|A||x|(i)$ $(x\in l_1,\ i=1,2,\cdots)$, $\||A|\|_1=\|A\|_1$, and $|A|^*=|A^*|$. We see also that $\||A|\|_p=\|A\|_p\leqq 1$ $(p=1,\infty)$ for each A in $\mathcal{S}'$. Let $\mathcal{S}_w$, $\mathcal{S}_w{}^*$, and $\mathcal{S}$ be subsets of $\mathcal{S}'$ that are defined by

$$\mathcal{S}_w=\{A\in\mathcal{S}' : |A|\in\mathcal{D}_w\},\quad \mathcal{S}_w{}^*=\{A\in\mathcal{S}' : |A|\in\mathcal{D}_w{}^*\},$$

$$\mathcal{S}=\{A\in\mathcal{S}' : |A|\in\mathcal{D}\}.$$

Define Q', Q_w, Q_w^*, and Q by

$$Q'=\{A\in\mathcal{S}' : |A|\in\mathcal{P}'\},\ Q_w=\{A\in\mathcal{S}_w : |A|\in\mathcal{P}_w\},$$

$$Q_w^*=\{A\in\mathcal{S}_w^* : |A|\in\mathcal{P}_w^*\},\ Q=\{A\in\mathcal{S} : |A|\in\mathcal{P}\}.$$

For each convex subset $\mathcal{C}$ of $\mathcal{S}'$, let ext $\mathcal{C}$ denote the set of extreme points of $\mathcal{C}$. An element A of $\mathcal{C}$ is called an extreme point (an extreme) of $\mathcal{C}$ if and only if $A=\frac{1}{2}(B+C)$ and $B, C\in\mathcal{C}$ imply $A=B=C$. Equivalently, $A\in$ ext $\mathcal{C}$ if and only if $B\in\mathcal{S}'$ and $A\pm B\in\mathcal{C}$ imply $B=0$. Suppose that $\mathcal{S}'$ is endowed with a topology τ and $\mathcal{E}\subset\mathcal{S}'$. The convex hull of $\mathcal{E}$ is denoted by ch $\mathcal{E}$ and the closed convex hull of $\mathcal{E}$ in the topology τ by cch$(\mathcal{E} : \tau)$.

In section 2, we shall give direct proofs to Theorem 1 : ext $\mathcal{D}_w=\mathcal{P}_w$ and Theorem 2 (Kendall and Kiefer): ext $\mathcal{D}'=\mathcal{P}'$. In Section 3, we shall prove (Theorems 3 and 4) that ext $\mathcal{S}'=Q_w\cup Q_w^*$ and $\mathcal{S}'=$cch $(Q : l_2$-w. o. t.). It is also shown (Theorems 5 and 6) that $\mathcal{S}_w^*\subsetneq$cch$(Q : l_1$-s. o. t.) and $\mathcal{S}\subsetneq$cch $(Q : l_1$-s. o. t.).

2. Extreme points of $\mathcal{D}_w$.

THEOREM 1. ext $D_w=\mathcal{P}_w$.

We see readily that $\mathcal{P}_w\subset$ext $\mathcal{D}_w$. It is therefore sufficient to show that $\mathcal{D}_w-\mathcal{P}_w\subset\mathcal{D}_w-$ext $\mathcal{D}_w$. Note that $\mathcal{D}_w-\mathcal{P}_w=(\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w)\cup(\mathcal{D}-\mathcal{P})$. It is known ([2], [5]) that $\mathcal{P}=$ext $\mathcal{D}$, so that $\mathcal{D}-\mathcal{P}\subset\mathcal{D}_w-$ext $\mathcal{D}_w$. Thus it remains to verify the following proposition.

PROPOSITION 1. $\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w\subset\mathcal{P}_w-$ ext $\mathcal{D}_w$.

Let $A=(a_{ij})\in\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w$. Denote by I the set of positive integers. Let $t_j=\Sigma_k a_{kj}(j\in I)$. Define $J_r\subset I$ $(r=0, 1, 2)$ by

$$J_0=\{j : t_j=0\},\ J_1=\{j : 0<t_j<1\},\ J_2=\{j : t_j=1\}.$$

Note that the sets J_0, J_1, and J_2 constitute a partition of the set I. If $J_1=\phi$, then $J_2\neq\phi$ and the matrix $A'=(a_{ij} : i\in I, j\in J_2)$ belongs to $\mathcal{D}-\mathcal{P}$, so that the matrix A is not an extreme of $\mathcal{D}_w$.

We now assume without loss of generality that $J_1\neq\phi$ and $0<a_{11}\leqq t_1<1$. We shall follow the terminology of Mauldon [5]. An ordered pair (i, j), where $i, j\in I$, is called a vertex of A if $a_{ij}>0$. Note that (1, 1) is a vertex of A by assumption. A finite collection of distinct vertices of A, $\{(i_r, j_r) : r=0, 1, \cdots, m\}$, is called a path in A (starting at the vertex (1, 1)) if

(i) $i_0=i_0=1$,

(ii) either $i_1=i_0$ or $j_1=j_0$,

(iii) if $i_{r-1}=i_r$, then $j_{r-1}\neq j_r=j_{r+1}$, and if $j_{r-1}=j_r$, then $i_{r-1}\neq i_r=i_{r+1}$.

Let K denote the union of all paths in A. Note that for each vertex (i, j) in K, there exists at least one path leading to the vertex (i, j). If there exist two distinct paths leading to the same vertex, there must exist a loop. By a loop we shall mean a finite collection of distinct vertices $\{(i_r, j_r) : r=0, 1, \cdots, 2n+1\}$ satisfying the conditions (ii) and (iii), together with the condition

(iv) $i_0=i_{2n+1}$ or $j_0=j_{2n+1}$.

LEMMA 1. *Suppose that* $A\in\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w$ *and* $0<a_{11}\leqq t_1<1$. *Then*

(i) *if there exist two distinct vertices* (k, m) *and* (k, n) *of* A *such that* $m, n\in J_1$, *then* A *is not an extreme of* $\mathcal{D}_w$;

(ii) *if there exists a loop in* A, *then* A *is not an extreme of* $\mathcal{D}_w$.

Proof. (i): Define the positive number b and the matrix $B=(b_{ij})$ by

$$b=\min\{a_{km}, a_{kn}, 1-t_m, 1-t_n\},$$

$$b_{km}=b, b_{kn}=-b, b_{ij}=0 \text{ elsewhere.}$$

Then $A\pm B\in\mathcal{D}_w$, so that A is not an extreme of $\mathcal{D}_w$.

(ii): Let $\{(i_r, j_r) : r=0, 1, \cdots, 2n+1\}$ be a loop in A such that $i_0\neq i_{2n+1}$ and $j_0=j_{2n+1}$. Define the positive number b and $B=(b_{ij})$ by

$$b=\min\{a_{i_r j_r} : r=0, 1, \cdots, 2n+1\}.$$

$$b_{i_r j_r}=(-1)^r b \quad (r=0, 1, \cdots, 2n+1), \quad b_{ij}=0 \text{ elsewhere.}$$

Then $A\pm B\in\mathcal{D}_w$, so that A is not an extreme of $\mathcal{D}_w$.

For each $A\in\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w$ with $0<a_{11}\leqq t_1<1$, define

$$T=m \quad (m=1, 2, \cdots)$$

if and only if there exist $q\in J_1$ with $q\neq 1$ and a path $\{(i_r, j_r) : r=0, 1, \cdots, m\}$ such that either

$$m=2n+1\,(n=0, 1, 2, \cdots), i_1=i_0(=1), \tag{3}$$

$$j_r\in J_2(r=1, \cdots, m-1), \quad j_m=q, \quad \text{or}$$

$$m=2n\,(n=1, 2, \cdots), \quad j_1=j_0(=1), \tag{4}$$

$$j_r\in J_2(r=2, \cdots, m-1), \quad j_m=q.$$

Otherwise, let $T=\infty$.

LEMMA 2. *Suppose that* $A\in\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w$ *and* $0<a_{11}\leqq t_1<1$. *If* $T<\infty$, *then* A *is not an extreme of* $\mathcal{D}_w$.

Proof. For $T\leqq 2$, A is not an extreme of $\mathcal{D}_w$ from Lemma 1(i). Suppose that $T\geqq 3$. If (3) holds, then the positive number b and the matrix $B=(b_{ij})$ are defined by

$$b=\min\{1-t_1, 1-t_q, a_{i_r j_r}\ (r=0, 1, \cdots, m)\}, \tag{5}$$

$$b_{i_r j_r}=(-1)^r b\,(r=0, 1, \cdots, m),\quad b_{ij}=0 \text{ elsewhere.} \tag{6}$$

If (4) holds b and $B=(b_{ij})$ are defined by (5) and (6) provided that $r=1, 2, \cdots, m$. In both cases, $A\pm B\in\mathcal{D}_w$, so that A is not an extreme of $\mathcal{D}_w$.

Proof of Proposition 1. Let $A=(a_{ij})\in\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w$ with $0<a_{11}\leqq t_1<1$. In view of preceding lemmas, we shall assume that for each $(i, j)\in K, t_j=1$ whenever $j\geqq 2$ and there exists a unique path leading to the vertex (i, j), and that every path can be indefinitely continued.

The following argument is a modification of Mauldon's argument [5, pp. 334, 335]. For each $(i, j)\in K$ with $(i, j)\neq(1, 1)$, let $p(i, j)$ denote the penultimate vertex of the unique path leading to the vertex (i, j). Define the matrix $D=(d_{ij})$ by

$$d_{ij}=\begin{cases} a_{ij} \text{ if } a_{ij}\leqq\frac{1}{2}, \\ & (i, j=1, 2, \cdots). \\ a_{ij}-1 \text{ if } a_{ij}>\frac{1}{2}. \end{cases}$$

For each $(i, j)\in K$ with $j\geqq 2$, let σ_{ij} denote the sum of the entries of the matrix D in the row or in the column containing (i, j) and $P(i, j)$. It is easily seen that $\sigma_{ij}=0$ or 1.

CASE 1. $\frac{1}{2}<t_1<1$. Define the real number w and the matrix (w_{ij}) by

$$w=(t_1-1)/\textstyle\sum_k d_{kl},$$

$$w_{ij}=\begin{cases} w & \text{for } (i, j)\in K, j=1, \\ \dfrac{d_{p(i,j)}}{d_{p(i,j)}-\sigma_{ij}} & \text{for } (i, j)\in K, j\geqq 2, \\ 0 & \text{elsewhere.} \end{cases}$$

If follows that $0<|w_{ij}|<\leqq 1$ for each $(i, j)\in K$. Define the matrix (m_{ij}) by

$$m_{ij}=\begin{cases} w & \text{for } (i,j)\in K, j=1, \\ w_{ij}m_{p(i,j)} & \text{for } (i,j)\in K, j\geqq 2, \\ 0 & \text{elsewhere.}\end{cases}$$

Note that $0<|m_{ij}|\leqq 1$ for each $(i,j)\in K$. Define the matrix $B=(b_{ij})$ by

$$b_{ij}=m_{ij}d_{ij}(i,j=1,2,\cdots).$$

Note that $b_{ij}=0$ for each $(i,j)\notin K$ and $0<|b_{ij}|\leqq|d_{ij}|\leqq a_{ij}$ for each $(i,j)\in K$. We have then $\sum_k b_{k1}=w\sum_k d_{k1}=t_1-1<0$, so that

$$0<\sum_k(a_{k1}+b_{k1})=2t_1-1<t_1<1,\ \sum_k(a_{k1}-b_{k1})=1.$$

We now show that $\sum_k b_{1k}=0$. It is easy to see that, since $(1,1)=p(1,k)$ for $(1,k)\in K$ with $k\geqq 2$,

$$b_{1k}=(-1)wd_{11}d_{1k}/\sum_{j(\neq 1)}d_{1j} \text{ for } (1,k)\in K,\ k\geqq 2, \text{ and } b_{11}=ud_{11}.$$

Thus, $\sum_{k(\neq 1)}b_{1k}=-b_{11}$. On the other hand, an argument of Mauldon [5, pp. 334, 335] shows that

$$\sum_k b_{ik}=0\,(i\geqq 2),\ \sum_k b_{kj}=0\ (j\geqq 2).$$

Thus, $A\pm B\in\mathcal{D}_w$, so that A is not an extreme of $\mathcal{D}_w$.

CASE 2. $0<t_1\leqq\frac{1}{2}$. Define the matrices (w_{ij}) and (m_{ij}) by

$$w_{ij}=\begin{cases} 1 & \text{for } (i,j)\in K,\ j=1, \\ \dfrac{d_{p(i,j)}}{d_{p(i,j)}-\sigma_{ij}} & \text{for } (i,j)\in K,\ j\geqq 2, \\ 0 & \text{elsewhere,}\end{cases}$$

$$m_{ij}=\begin{cases} 1 & \text{for } (i,j)\in K,\ j=1, \\ w_{ij}m_{p(i,j)} & \text{for } (i,j)\in K,\ j\geqq 2, \\ 0 & \text{elsewhere.}\end{cases}$$

Define the matrix $B=(b_{ij})$ by $b_{ij}=m_{ij}d_{ij}(i,j=1,2,\cdots)$. Note that $b_{ij}=0$ for each $(i,j)\notin K$ and $0<|b_{ij}|\leqq|d_{ij}|\leqq a_{ij}$ for each $(i,j)\in K$. We see easily that $b_{k1}=d_{k1}=a_{k1}$ $(k=1,2,\cdots)$ and $\sum_k b_{k1}=t_1$, so that

$$0<\sum_k(a_{k1}+b_{k1})=2t_1\leqq 1,\ \sum_k(a_{k1}-b_{l1})=0.$$

As in Case 1, we also obtain that

$$\sum_k b_{kj}=0\ (j\geqq 2),\ \sum_k b_{ik}=0\ (i\geqq 1).$$

Thus, $A\pm B\in\mathcal{D}_w$, so that A is not an extreme of $\mathcal{D}_w$.

COROLLARY 1. ext $\mathcal{D}_w^*=\mathcal{P}_w^*$.

A nonnegative matrix $A=(a_{ij} : i=1, \cdots, m, \quad j=1, 2, \cdots)$, where m is a positive integer, is called an (m, ∞)-w. d. s. matrix if $\sum^{\infty}_{k=1} a_{ik}=1 \quad (i=1, 2, \cdots, m)$ and $\sum^{m}_{k=1} a_{kj}\geqq 1 \quad (j=1, 2, \cdots)$ Denote by $\mathcal{D}_w(m, \infty)$. the convex set of (m, ∞)-w. d. s. matrices. Let $\mathcal{P}_w(m, \infty)=\{(a_{ij})\in\mathcal{D}_w(m, \infty) : \ a_{ij}=0$ or $1 (i=1, 2, \cdots, m, \ j=1, 2, \cdots)\}$.

PROPOSITION 2. ext $\mathcal{D}_w(m, \infty)=\mathcal{P}_w(m, \infty)$.

The proof follows from the proofs of Lemmas 1 and 2, together with the fact that each path in (m, ∞)-w. d. s. matrix has a finite length, or yields a loop.

Let $\mathcal{D}_w^*(\infty, m)$ and $\mathcal{P}_w^*(\infty, m)$ denote the convex set of transposes of matrices in $\mathcal{D}_w(m, \infty)$ and the set of transposes of matrices in $\mathcal{P}_w(m, \infty)$.

COROLLARY 2. ext $D_w^*(\infty, m)=\mathcal{P}_w^*(\infty, m)$.

REMARK. The foregoing arguments also show that for each $A=(a_{ij})$ in $\mathcal{D}_w-\mathcal{D}-\mathcal{P}_w(\mathcal{D}_w(m, \infty)-\mathcal{P}_w(m, \infty))$, there exists a nonzero matrix $B=(b_{ij})$ such that $\sum_k b_{ik}=0$ for each i, $\sum_k b_{kj}=0$ for each $j\in J_2$, and $A\pm B\in\mathcal{D}_w$ $(\mathcal{D}_w(m, \infty))$.

THEOREM 2. (*Kendall-Kiefer*) ext $\mathcal{D}'=\mathcal{P}'$.

Proof. Since $\mathcal{P}'\subset$ext $\mathcal{D}'$, we shall show that $\mathcal{D}'-\mathcal{P}'\subset\mathcal{D}'-$ext $\mathcal{D}'$. Note that $\mathcal{D}'-\mathcal{P}'=(\mathcal{D}'-(\mathcal{D}_w\cup\mathcal{D}_w^*\cup\mathcal{P}'))\cup(\mathcal{D}_w\cup\mathcal{D}_w^*-\mathcal{P}')$ and, by Theorem 1 and Corollary 1, $\mathcal{D}_w\cup\mathcal{D}_w^*-\mathcal{P}'\subset\mathcal{D}'-$ext $\mathcal{D}'$. Thus it suffices to show that $\mathcal{D}'-(\mathcal{D}_w\cup\mathcal{D}_w^*\cup\mathcal{P}')\subset\mathcal{D}'-$ext $\mathcal{D}'$. Let $A=(a_{ij})\in\mathcal{D}'-(\mathcal{D}_w\cup\mathcal{D}_w^*\cup\mathcal{P}')$. Let $s_i=\Sigma_k a_{ik}(i\in I)$. Define $I_r\subset I(r=0, 1, 2)$ by

$$I_0=\{i : s_i=0\}, \quad I_1=\{i : 0<s_i<1\}, \quad I_2=\{i : s_i=1\}.$$

Let $K'=\{(i, j) : 0<a_{ij}<1\}$. Evidently K' is non-empty and is the union of sets $K'\cap(I_i\times J_j) \quad (i, j=1, 2)$.

CASE 1. $K'\cap(I_1\times J_1)\neq\phi$. Select a vertex (p, q) from $K'\cap(I_1\times J_1)$, so that $0<a_{pq}<s_p$, $t_q<1$. Define the positive number b and the nonzero matrix $B=(b_{ij})$ by

$$b=\min\{a_{pq}, \ 1-s_p, \ 1-t_q\},$$

$$b_{pq}=b, \ b_{ij}=0 \text{ elsewhere.}$$

Then $A\pm B\in\mathcal{D}'$, so that A is not an extreme of $\mathcal{D}'$.

CASE 2. $K'\cap(I_1\times J_1)=\phi$, $K'\cap(I_2\times J_1)\neq\phi$. Define $J_2=\{j\in J_2 : \sum_{k\in I_2}a_{kj}=1\}$. Suppose that $J_2'=\phi$. Pick a vertex (m, n) in $K'\cap(I_2\times J_1)$. Then there exists p in I such that $n\neq p$ and $0<a_{mp}<1$. If $p\in J_1$, then by Lemma

1(i), the matrix A is not an extreme. If $p\in J_2$, then there exists $q\in I_1$ such that $0<a_{qp}<1$. Define the positive number b and the matrix $B=(b_{ij})$ by

$$b=\min\{a_{mn}, a_{mp}, a_{qp}, 1-t_n, 1-s_q\},$$

$$b_{mn}=b_{qp}=b, \quad b_{mp}=-b, \quad b_{ij}=0 \text{ elsewhere}.$$

Then, $A\pm B\in\mathcal{D}'$, so that A is not an extreme.

If $J_2'\neq\phi$, then define the matrix A' by

$$A'=(a_{ij} : i\in I_2, j\in I).$$

We see that A' is in $\mathcal{D}_w-\mathcal{P}_w$ or in $\mathcal{D}_w(m,\infty)-\mathcal{P}_w(m,\infty)$ according as the set I_2 is infinite or finite. By Remark, there exists a nonzero matrix $B'=(b'_{ij} : i\in I_2, j\in I)$ such that $A'\pm B'\in\mathcal{D}_w$ and $\sum_{k\in I_2}b'_{ij}=0$ for each $j\in J_2'$. Define the matrix $B=(b_{ij} : i,j\in I)$ by

$$b_{ij}=b'_{ij} \quad (i\in I_2, \ j\in I), \quad b_{ij}=0 \text{ elsewhere}.$$

Then $A\pm B\in\mathcal{D}'$, so that A is not an extreme of $\mathcal{D}'$.

CASE 3. $K'\cap(I\times J_1)=\phi$, or equivalently $K'\subset I\times J_2$. Define the matrix A' by

$$A'=(a_{ij} : i\in I, \quad j\in J_2).$$

Then $A'\in\mathcal{D}_w^*-\mathcal{P}_w^*$ or $A'\in\mathcal{D}_w^*(\infty,m)-\mathcal{P}_w^*(\infty,m)$ according as the set J_2 is infinite or finite. It follows from either Corollary 1 or Corollary 2 that there exist two distinct matrices $B'=(b'_{ij} : i\in I, j\in J_2)$ and $C'=(c'_{ij} : i\in I, j\in J_2)$ in $\mathcal{D}_w^*$ (or in $\mathcal{D}_w^*(\infty,m)$) such that $A'=\frac{1}{2}(B'+C')$. Define w*. d. s. matrices $B=(b_{ij} : i,j\in I)$ and $C=(c_{ij} : i,j\in I)$ by,

$$b_{ij}=b'_{ij}(i\in I, \ j\in J_2), \quad b_{ij}=0 \text{ elsewhere},$$

$$c_{ij}=c'_{ij} \quad (i\in I, \ j\in J_2), \quad c_{ij}=0 \text{ elsewhere}.$$

Since $a_{ij}=0$ for each (i,j) in $I\times(J_0\cap J_1)$, we have that $A=\frac{1}{2}(B+C)$, $B\neq C$, so that A is not an extreme of $\mathcal{D}'$.

3. Extreme points of $\mathcal{S}'$ and approximation theorems.

We shall begin this section with the following theorem.

THEOREM 3. ext $\mathcal{S}'=Q_w\cup Q_w^*$.

Proof. Suppose that $A=(a_{ij})\in Q_w$, so that $|A|=(|a_{ij}|)\in\mathcal{P}_w$. Then there exist a unique partition (E_1, E_2) of the set I of positive integers and an injection $\varphi : I\to I$ such that for each $i \in E_1$, $a_{ij} = \delta_{\varphi(i), j}$ $(j = 1, 2, \cdots)$ and

for each $i \in E_2$, $a_{ij} = -\delta_{\varphi(i), j}$ $(j = 1, 2, \cdots)$.

Assume that $A = \frac{1}{2}(B + C)$, where $B = (b_{ij})$, $C = (c_{ij}) \in \mathcal{S}'$. For each $i \in E_1$, since $\sum_j |b_{ij}| \leqq 1$, $\sum_j |c_{ij}| \leqq 1$, and

$$\delta_{\varphi(i), j} = \frac{1}{2}(b_{ij} + c_{ij}) \quad (j = 1, 2, \cdots),$$

we must have $\delta_{\varphi(i), j} = b_{ij} = c_{ij}$ $(j = 1, 2, \cdots)$. Similarly, we also have, for each $i \in E_2$, $-\delta_{\varphi(i), j} = b_{ij} = c_{ij}$ $(j = 1, 2, \cdots)$. This shows that $A = B = C$ and $Q_w \subset \operatorname{ext} \mathcal{S}'$.

For each $A = (a_{ij}) \in Q_w{}^*$, there exist a partition (F_1, F_2) of the set I and an injection $\psi : I \to I$ such that

for each $j \in F_1$, $a_{ij} = \delta_{i, \psi(j)}$ $(i = 1, 2, \cdots)$ and

for each $j \in F_2$, $a_{ij} = -\delta_{i, \psi(j)}$ $(i = 1, 2, \cdots)$.

We see readily that $A \in \operatorname{ext} \mathcal{S}'$. Thus, $Q_w \cup Q_w{}^* \subset \operatorname{ext} \mathcal{S}'$.

It remains to show that $\mathcal{S}' - (Q_w \cup Q_w{}^*) \subset \mathcal{S}' - \operatorname{ext} \mathcal{S}'$. Note that $\mathcal{S}' - (Q_w \cup Q_w{}^*) = (\mathcal{S}' - Q') \cup (Q' - (Q_w \cup Q_w{}^*))$. If $A = (a_{ij}) \in \mathcal{S}' - Q'$, then, by Theorem 2, $|A| \in \mathcal{D}' - \mathcal{P}' = \mathcal{D}' - \operatorname{ext} \mathcal{D}'$, so that

$$|A| = \frac{1}{2}(B' + C'), \text{ where } B', C' \in \mathcal{D}' \text{ and } B' \neq C'.$$

Let $B' = (b'_{ij})$ and $C' = (c'_{ij})$. Then

$$|a_{ij}| = \frac{1}{2}(b'_{ij} + c'_{ij}) \quad (0 \leqq b'_{ij},\ c'_{ij} \leqq 1,\ i, j = 1, 2, \cdots).$$

Define the matrices $B = (b_{ij})$ and $C = (c_{ij})$ by

$$b_{ij} = \operatorname{sgn}(a_{ij}) b'_{ij}, \quad c_{ij} = \operatorname{sgn}(a_{ij}) c'_{ij}.$$

Clearly, $B, C \in \mathcal{S}'$, $B \neq C$, and $A = \frac{1}{2}(B + C)$, so that A is not an extreme of $\mathcal{S}'$.

Suppose now that $A = (a_{ij}) \in Q' - (Q_w \cup Q_w{}^*)$. Then $|A| \in \mathcal{P}' - (\mathcal{P}_w \cup \mathcal{P}_w{}^*)$, so that we may assume without loss of generality that

$$a_{1j} = 0 \quad (j = 1, 2, \cdots) \text{ and } a_{i1} = 0 \quad (i = 1, 2, \cdots).$$

Define the matrices $B = (b_{ij})$ and $C = (c_{ij})$ by

$$b_{11} = 1,\ b_{1j} = 0 \ (j = 2, 3, \cdots),\ b_{i1} = 0 \ (i = 2, 3, \cdots),$$

$$b_{ij} = a_{ij} \text{ elsewhere},$$

$$c_{11}=-1, c_{1j}=0 \ (j=2,3,\cdots), \ c_{i1}=0 \ (i=2,3,\cdots),$$
$$c_{ij}=a_{ij} \text{ elsewhere.}$$

It is evident that $B, C\in Q'$, $B\neq C$, and $A=\frac{1}{2}(B+C)$, so that A is not an extreme of S'. This completes the proof.

We shall state and prove an analogue of Theorem 3 for finite matrices. Let n denote a positive integer. Let $\mathcal{D}'(n)$ and $\mathcal{D}(n)$ denote the convex set of $n\times n$-d. s. s. matrices and the convex set of $n\times n$-d. s. s. matrices. Note that for each $(a_{ij})\in\mathcal{D}'(n)$, $(a_{ij})\in\mathcal{D}(n)$ iff $\sum_k a_{ik}=1$ for each i iff $\sum_k a_{kj}=1$ for each j. Denote by $\mathcal{P}'(n)$ the set of those matrices (a_{ij}) in $\mathcal{D}'(n)$ such that $a_{ij}=0$ or 1 $(i,j=1,2,\cdots,n)$. Let $\mathcal{P}$ (n) denote the set of $n\times n$-permutation matrices. Let $\mathcal{S}'(n)$ be the convex set of $n\times n$-(real) matrices (a_{ij}) such that $(|a_{ij}|)\in\mathcal{D}'(n)$ and let $\mathcal{S}(n)$ be the set of those matrices (a_{ij}) in $\mathcal{S}'(n)$ such that $(|a_{ij}|)\in\mathcal{D}(n)$. Note that $\mathcal{S}(n)$ is not convex. Define $Q'(n)$ and $Q(n)$ by

$$Q'(n)=\{(a_{ij})\in\mathcal{S}'(n) : (|a_{ij}|)\in\mathcal{P}'(n)\} \text{ and}$$
$$Q(n)=\{(a_{ij})\in\mathcal{S}(n) : (|a_{ij}|)\in\mathcal{P}(n)\}.$$

PROPOSITION 3. $\mathrm{ext}\mathcal{S}'(n)=Q(n)$, $\mathcal{S}'(n)=\mathrm{ch}Q(n)$.

Proof. By using the method of proof of Theorem 3, together with simple arguments, we see that ext $\mathcal{S}'(n)=Q(n)$ and $Q'(n)\subset\mathrm{ch}\, Q(n)$. To prove that $\mathcal{S}'(n)\subset\mathrm{ch}\, Q(n)$, it suffices to prove that $\mathcal{S}'(n)\subset\mathrm{ch}\, Q'(n)$. For each $A=(a_{ij})$ in $\mathcal{S}'(n)$, we have $(|a_{ij}|)\in\mathcal{D}'(n)$ and, since $\mathcal{D}'(n)=\mathrm{ch}\,\mathcal{P}'(n)$ [4, Lemma F],

$$(|a_{ij}|)=\textstyle\sum_{t=1}^{r}c_t(P_{ij}^t) \ (0\leqq c_1,\cdots,c_r\leqq 1, \ \sum_{t=1}^{r}c_t=1;$$
$$(P_{ji}^{\ t})\in\mathcal{P}'(n), \ t=1,\cdots,r).$$

Define $Q_t=(q_{ij}^{\ t})$ $(t=1,2,\cdots,r)$ by $q_{ij}^{\ t}=\mathrm{sgn}(a_{ij})p_{ij}^{\ t}$. It follows that $Q_t\in Q'(n)$ $(t=1,\cdots,r)$ and $A=\sum_{t=1}^{r}c_tQ_t\in\mathrm{ch}\, Q'(n)\subset\mathrm{ch}\, Q(n)$. Evidently $\mathrm{ch}\, Q(n)\subset\mathcal{S}'(n)$, so that the proof is complete.

Let $\mathcal{Y}$ denote the Cartesian product of countably infinite copies of the real line with the Tychonoff topology (the topology of simple convergence), and let $\mathcal{Z}$ be the Cartesian product of countably infinite copies of the interval $[-1,1]$. It is easy to see that $\mathcal{Y}$ is a Fréchet space (a complete metric vector space) in which $\mathcal{Z}$ is compact. By means of elementary arguments, we may verify that $\mathcal{S}'(\subset\mathcal{Z})$ is a compact convex subset of $\mathcal{Y}$. On the other hand, it is straightforward to show that $\mathcal{S}'$ as a subset of $[l_2]$ with the l_2-w. o. t. is compact, and that on $\mathcal{S}'$, the induced l_2-w. o. t. and the induced Tychonff topology coincide. Thus we obtain the following lemma.

LEMMA 3. *$\mathcal{S}'$ is a compact convex subset of $\mathcal{Y}$ in the l_2-w. o. t., or equivalently in the Tychonoff topology.*

THEOREM 4. $\mathcal{S}'=\mathrm{cch}(Q : l_2\text{-w. o. t.})$

Proof. In view of Lemma 3, it is enough to show that for each $A=(a_{ij})$ in $\mathcal{S}'$ and for each positive integer n, there exists $B=(b_{ij})\in\mathrm{ch}\ Q$ such that $a_{ij}=b_{ij}$ $(i, j=1, 2, \cdots, n)$. Define $A_n=(a_{ij}')\in\mathcal{S}'(n)$ by $a_{ij}'=a_{ij}$ $(i, j=1, 2, \cdots, n)$. We have from Proposition 3 that

$$A_n=\sum_{t=1}^{r}c_tB_{nt}\ (0\leqq c_1, \cdots, c_r\leqq 1,\ \ \sum_{t=1}^{r}=1\ ;\ \ B_{nt}\in Q(n),\ \ t=1, \cdots, r).$$

Extend each matrix in $Q(n)$ to a matrix B_t in Q. For example, if $B_{nt}=(b_{ij}')$ then we may define $B_t=(b_{ij})$ by

$$a_{ij}=b_{ip}'(i, j=1, 2, \cdots, n),\ \ b_{ij}=\delta_{ij}\ \ (i, j=n+1,\ \ n+2, \cdots),$$
$$b_{ij}=0 \text{ elsewhere.}$$

Define the matrix B by $B=\sum_{t=1}^{r}c_tB_t$. Clearly the matrix B has the desired property, and the proof is complete.

The converse of the Krein-Milman theorem [1, p. 440] then shows that ext $\mathcal{S}'\subset\mathrm{cl}(Q : l_2\text{-w. o. t.})$, the closure of Q in the l_2-w. o. t. We may easily verify that Q' is closed in the l_2-w. o. t., and that $Q'=\mathrm{cl}(Q : l_2\text{-w. o. t.})$, so that ext $\mathcal{S}'\subset Q'$. Thus, Theorem 4, together with the converse of the Krein-Milman theorem, does not lead to Theorem 3. Since ch$Q$ has the same closure in the weak operator topology and in the strong operator topology for $[l_2]$ [1, p. 477], we also obtain $\mathcal{S}'=\mathrm{cch}(Q : l_2\text{-s. o. t.})$.

THEOREM 5. $\mathcal{S}_w^*\subsetneqq\mathrm{cch}(Q : l_1\text{-s. o. t.})$.

Proof. It is easy to see that $\mathcal{S}_w^*$ is not convex and closed in the l_1-s. o. t. Let $A=(a_{ij})\in\mathcal{S}_w^*$. Note that $|A|=(|a_{ij}|)\in\mathcal{D}_w^*$. Since $\mathcal{D}_w^*=\mathrm{cch}(\mathcal{P} : l_1$-s. o. t. [3, p. 87, Remark], there exists, for each $\varepsilon>0$ and for each positive integer n, a matrix $B=(b_{ij})\in\mathrm{ch}\ \mathcal{P}$ such that $\sum_k||a_{kj}|-b_{kj}|<\varepsilon(j=1, 2, \cdots n)$. Suppose that $B=\sum_{t=1}^{r}c_t\ P_t\ \in\mathrm{ch}\mathcal{P}$, where $P_t=(p_{ij}^t)$, $t=1, 2, \cdots, r$. Define $Q_t=(q_{ij}^t)\in Q(t=1, 2, \cdots, r)$ by $q_{ij}^t=\mathrm{sgn}(a_{ij})p_{ij}^t)\ (i, j=1, 2, \cdots)$ and $C=\sum_{t=1}^{r}c_tQ_t$. It follows that $C=(c_{ij})\in\mathrm{ch}\ Q$ and

$$\sum_k|a_{kj}-c_{kj}|=\Sigma_k||a_{kj}|-b_{kj}|<\varepsilon\ \ (j=1, 2, \cdots, n).$$

This completes the proof.

On the set $\mathcal{S}'$, we may define a topology induced by ε-neighbourhoods of the form

$$\{(b_{ij})\ :\ \sum_k|a_{ik}-b_{ik}|<\varepsilon,\ \ i=1, 2, \cdots, n\}.$$

As an immediate consequence of Theorem 5, we see that $\mathcal{S}_w$ is a closed (proper) subset of the closed convex hull of Q in the topology mentioned above.

THEOREM 6. $\mathcal{S} \subsetneqq \mathrm{cch}(Q : l_1\text{-s}^*.\text{o.t.})$.

Proof. It is easily seen that $\mathcal{S}$ is a closed set in the l_1-s*. o. t. Let $A=(a_{ij})\in\mathcal{S}$. Then $|A|=(|a_{ij}|)\in\mathcal{D}$, so that by a theorem of Rattray and Peck ([6], [3, p. 89]) : $\mathcal{D}=\mathrm{cch}(Q : l_1\text{-s}^*\text{o.t.})$, there exists, for $\varepsilon>0$ and for each positive integer n, a matrix $B=(b_{ij})\in\mathrm{ch}\,\mathcal{D}$ such that

$$\textstyle\sum_k \big||a_{ik}|-b_{ik}\big|<\varepsilon,\quad \sum_k \big||a_{kj}|-b_{kj}\big|<\varepsilon\,(i,j=1,\cdots,n).$$

By using the method of proof Theorem 5, we can find a matrix $C=(c_{ij})$ in ch Q such that

$$\textstyle\sum_k |a_{ik}-c_{ik}|<\varepsilon,\quad \sum_k |a_{kj}-c_{kj}|<\varepsilon\,(i,j=1,\cdots,n).$$

This completes the proof.

Bibliography

[1] Dunford, N., Schwartz, J. T.: *Linear operators,* Part I. New York: Interscience, 1967.

[2] Kendall, D. G.: *On infinite doubly stochastic matrices and Birkhoff's problem,* 111., J. London math. Spc. **35**, 81-84 (1960).

[3] Kim, C. W.: *A pproximations of doubly substochastic operators,* Michgan Math. J. **19**, 83-95 (1972).

[4] Kim, C. W.: *Approximations of positive contractions on L^∞,* II, Math. Ann. (to apear)

[5] Mauldon, J. G.: *Extreme points of convex sets of doubly stochastic matrices,* I, Z, Wahrscheinlichkeitstheorie verw. Geb. **13**, 333-337 (1969).

[6] Rattray, B. A., Peck, J. E. L.: *Infinite stochastic matrices.* Trans. Roy. Soc. Canada, Sect. III (3) **49**, 55-57 (1955).

Simon Fraser University, Canada