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ON INFINITESIMAL MINIMAL VARIATIONS OF ANTI-INVARIANT SUBMANIFOLDS OF A KAEHLERIAN MANIFOLD

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Introduction

Chen ([2]), Yano ([2], [3], [5], [9]), Pak ([9]) and one of the present authors ([9]) have recently studied infinitesimal variations of submanifolds of a Riemannian manifold.

On the other hand, Yano and Kon ([8]) have studied infinitesimal anti-invariant normal variations of submanifolds M^m of a Kaehlerian manifold M^{2m}.

The main purpose of the present paper is to investigate infinitesimal anti-invariant variations which carry a minimal submanifold into a minimal submanifold. Such an infinitesimal variation will be called in this paper an anti -invariant minimal variation.

In §1, we prepare the structure equations of the anti invariant submanifold of a Kaehlerian manifold. In §2 we recall the fundamental properties of anti-invariant variations obtained in [8] and review the definitions of parallel and isometric variations. In §3, we derive integral formulas on isometric variations and prove some theorems to be parallel. In the last §4 we study anti-invariant minimal variations which preserve $f_b{}^x$ of the submanifold of a Kaehlerian manifold of constant holomorphic sectional curvature.

§1. Anti-invariant submanifolds of a Kaehlerian manifold.

Let M^{2m} be a real $2m$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U;\ x^h\}$ and with almost complex structure tensor $F_i{}^h$ and Hermitian metric tensor g_{ji}, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 2m\}$. Then we have

$$(1.1)\qquad F_i{}^t F_t{}^h = -\delta_i^h,\quad F_j{}^t F_i{}^s g_{ts} = g_{ji},\quad \nabla_j F_i{}^h = 0,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbols Γ_{ji}^h formed with g_{ji}.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V;\ y^a\}$ and with metric tensor g_{cb}, where, here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, n\}$. We

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assume that M^n is isometrically immersed in M^{2m} by the immersion $i: M^n \to M^{2m}$ and identify $i(M^n)$ with M^n. We represent the immersion $i: M^n \to M^{2m}$ locally by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h$, $(\partial_b = \partial/\partial y^b)$, which are n linearly independent vectors of M^{2m} tangent to M^n. Since the immersion i is isometric, we have

$$(1.2) \qquad g_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

We denote by $C_y{}^h$ $2m-n$ mutually orthogonal unit normals to M^n, where, here and in the sequel, the indices $x, y, z, \cdots$ run over the range $\{n+1, n+2, \cdots, 2m\}$.

Then the equations of Gauss are written as

$$(1.3) \qquad \nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h,$$

where ∇_c denotes the operator of van der Waerden-Bortolotti covariant differentiation along M^n and $h_{cb}{}^x$ are second fundamental tensors of M^n with respect to the normals $C_x{}^h$ and those of Weingarten

$$(1.4) \qquad \nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where $h_c{}^a{}_y = h_{cby} g^{ba} = h_{cb}{}^z g^{ba} g_{zy}$, g^{ba} denoting contravariant components of the metric tensor g_{cb} of M^n, and g_{zy} the metric tensor of the normal bundle.

If the transform by F of any vector tangent to M^n is always normal to M^n, that is, if there exists a tensor field $f_b{}^x$ of mixed type such that

$$(1.5) \qquad F_i{}^h B_b{}^i = -f_b{}^x C_x{}^h,$$

we say that M^n is *anti-invariant* (or *totally real*) in M^{2m} ([6], [7]).

For the transform by F of normal vectors $C_y{}^h$, we have equations of the form

$$(1.6) \qquad F_i{}^h C_y{}^i = f_y{}^a B_a{}^h + f_y{}^x C_x{}^h,$$

where $f_y{}^a = f_b{}^z g^{ba} g_{zy}$, which can also be written as $f_{ya} = f_{ay}$, where $f_{ya} = f_y{}^b g_{ba}$ and $f_{ay} = f_a{}^z g_{zy}$.

From (1.5) and (1.6) we find

$$(1.7) \qquad f_b{}^y f_y{}^a = \delta_b{}^a,$$

$$(1.8) \qquad f_b{}^y f_y{}^x = 0, \qquad f_y{}^z f_z{}^a = 0,$$

$$(1.9) \qquad f_y{}^z f_z{}^x = -\delta_y{}^x + f_y{}^a f_a{}^x.$$

Differentiating (1.5) and (1.6) covariantly along M^n, and using equations of Gauss and Weingarten, we find

$$(1.10) \qquad h_{cb}{}^x f_x{}^a - h_c{}^a{}_x f_b{}^x = 0,$$

$$(1.11) \qquad \nabla_c f_b{}^x = -h_{cb}{}^y f_y{}^x, \quad \nabla_c f_y{}^a = h_c{}^a{}_x f_y{}^x,$$

(1.12) $$\nabla_c f_y{}^x = h_c{}^a{}_y f_a{}^x - h_{ca}{}^x f_y{}^a.$$

Equations of Gauss, Codazzi and Ricci are respectively

(1.13) $$K_{dcb}{}^a = K_{kji}{}^h B_d{}^k{}_c{}^j{}_b{}^i{}^a{}_h + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$$

(1.14) $$0 = K_{kji}{}^h B_d{}^k{}_c{}^j{}_b{}^i C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x)$$

and

(1.15) $$K_{dcy}{}^x = K_{kji}{}^h B_d{}^k{}_c{}^j C_y{}^i C^x{}_h + (h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y)$$

where

$$B_d{}^k{}_c{}^j{}_b{}^i{}^a{}_h = B_d{}^k B_c{}^j B_b{}^i B^a{}_h, \qquad B_d{}^k{}_c{}^j{}_b{}^i = B_d{}^k B_c{}^j B_b{}^i, \qquad B_d{}^k{}_c{}^j = B_d{}^k B_c{}^j,$$

$$C^x{}_h = C_y{}^i g^{yx} g_{ih},$$

$K_{dcy}{}^x$ being the curvature tensor of the connection induced in the normal bundle.

§ 2. Infinitesimal variations of anti-invariant submanifolds. ([8])

We consider an infinitesimal variation of anti-invariant submanifold M^n of a Kaehlerian manifold M^{2m} given by

(2.1) $$\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon,$$

where $\xi^h(y)$ is a vector field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

(2.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallelly from the varied point $(\bar{x}^h)$ to the original point (x^h). We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h (x + \xi\varepsilon) \xi^j \bar{B}_b{}^i \varepsilon$$

at the point (x^h), or

(2.3) $$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon$$

neglecting the terms of order higher than one with respect to ε, where

(2.4) $$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i.$$

In the sequel we always neglect terms of order higher than one with respect to ε. Thus putting

(2.5) $$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h,$$

we have from (2.3)

(2.6) $$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon.$$

Putting

(2.7) $$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h,$$

we have

(2.8) $$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h.$$

Now we denote by $\bar{C}_y{}^h$ 2m−n mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_c{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

(2.9) $$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma_{ji}{}^h (x + \xi\varepsilon) \xi^j \bar{C}_y{}^i \varepsilon.$$

We put

(2.10) $$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h$$

and assume that $\delta C_y{}^h$ is of the form

(2.11) $$\delta C_y{}^h = \eta_y{}^h \varepsilon = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Then, from (2.6), (2.10) and (2.11), we have

(2.12) $$\bar{C}_y{}^h = C_y{}^h - \Gamma_{ji}{}^h \xi^j C_z{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Applying the operator δ to $B_b{}^j C_y{}^i g_{ji} = 0$ and using (2.6), (2.8), (2.11) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi_y + h_{bay} \xi^a) + \eta_{yb} = 0,$$

where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, or

(2.13) $$\eta_y{}^a = -(\nabla^a \xi_y + h_b{}^a{}_y \xi^b),$$

∇^a being defined to be $\nabla^a = g^{ac}\nabla_c$. Applying the operator δ to $C_y{}^j C_x{}^i g_{ji} = \delta_{yx}$ and using (2.11) and $\delta g_{ji} = 0$, we find

(2.14) $$\eta_{yx} + \eta_{xy} = 0,$$

where $\eta_{yx} = \eta_y{}^z g_{zx}$.

An infinitesimal variation given by (2.1) is called an *anti-invariant variation* if it carries an anti-invariant submanifold into an anti-invariant submanifold. For an infinitesimal variation given by (2.1), when $\xi^x = 0$, that is, when the variation vector ξ^h is tangent to the submanifold we say that the variation is *tangential* and when $\xi^a = 0$, that is, when the variation vector ξ^h is normal to the submanifold we say that the variation is *normal*.

Yano and Kon have proved in their paper ([8]):

THEOREM 2.1. *In order for an infinitesimal variation to carry an anti-*

invariant submanifold into an anti-invariant submanifold, it is necessary and sufficient that the variation vector ξ^h *satisfies*

$$(\nabla_b \xi^x) f_y{}^a = f_b{}^x (\nabla^a \xi_x). \tag{2.15}$$

In this case the variation of $f_b{}^x$ *is given by*

$$\delta f_b{}^x = [(\nabla_b \xi^a - h_b{}^a{}_y \xi^y) f_a{}^x - (\nabla_b \xi^y + h_{ba}{}^y \xi^a) f_y{}^x - f_b{}^y \eta_z{}^x] \varepsilon. \tag{2.16}$$

Suppose that an infinitesimal variation $\bar{x}^h = x^h + \xi^h \varepsilon$ carries a submanifold $x^h = x^h(y)$ into another submanifold $\bar{x}^h = \bar{x}^h(y)$ and the tangent space of the original submanifold at (x^h) and that of the varied sumanifold at the corresponding point $(\bar{x}^h)$ are parallel. Then we say that the variation is *parallel* ([5]). Since we have from (2.5), (2.6) and (2.8)

$$\tilde{B}_b{}^h - [\delta_b^a + (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) \varepsilon] B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h \varepsilon,$$

we have

LEMMA 2.2 ([5]). (1) *In order for an infinitesimal variation to be parallel, it is necessary and sufficient that*

$$\nabla_b \xi^x + h_{ba}{}^x \xi^a = 0. \tag{2.17}$$

(2) *A parallel variation is an anti-invariant variation.*

Applying the operator δ to (1.5) and using (2.6), (2.8) and $\delta g_{ji} = 0$, we find (cf. [6])

$$\delta g_{cb} = (\nabla_{bc} \xi + \nabla_b \xi_c - 2 h_{cbx} \xi^x) \varepsilon, \tag{2.18}$$

from which

$$\delta g^{ba} = -(\nabla^b \xi^a + \nabla^a \xi^b - 2 h^{ba}{}_x \xi^x) \varepsilon. \tag{2.19}$$

A variation of a submanifold for which $\delta g_{cb} = 0$ is said to be *isometric.*

Now we assume that an anti-invariant variation preserves $f_b{}^x$, that is, $\delta f_b{}^x = 0$. Then (1.7), (1.8) and (2.16) imply

$$\nabla_b \xi_c - h_{bcy} \xi^y = f_b{}^y f_c{}^x \eta_{yx}. \tag{2.20}$$

Thus, by (2.14), (2.18) and (2.20), we have $\delta g_{cb} = 0$. Therefore we obtain

THEOREM 2.3 ([8]). *If an anti-invariant variation preserves* $f_b{}^x$, *then the variation is isometric.*

§ 3. Isometric variations on compact submanifolds.

In this section we compute infinitesimal variations of the second fundamental tensors and Christoffel symbols, and prove some integral formulas on an isometric variation.

For a tensor field carrying three kinds of indices, say $T_{by}{}^{h}$, it is well known that ([5])

(3.1) $$\delta\nabla_c T_{by}{}^{h}-\nabla_c\delta T_{by}{}^{h} = K_{kji}{}^{h}\xi^{k}B_c{}^{j}T_{by}{}^{i}\varepsilon-(\delta\Gamma_{cb}{}^{a})T_{ay}{}^{h}-(\delta\Gamma_{cy}{}^{x})T_{bx}{}^{h},$$

$\delta\Gamma_{cb}{}^{a}$ and $\delta\Gamma_{cy}{}^{x}$ being the variation of the affine connection $\Gamma_{cb}{}^{a}$ induced on M^n and that of the affine connection induced on the normal bundle of M^n respectively. Applying formula (3.1) to $B_b{}^{h}$, we find

$$\delta\nabla_c B_b{}^{h}-\nabla_c\delta B_b{}^{h}=K_{kji}{}^{h}\xi^{k}B_c{}^{j}B_b{}^{i}\varepsilon-(\delta\Gamma_{cb}{}^{a})B_a{}^{h},$$

or using (1.3) and (2.6)

$$\delta(h_{cb}{}^{x}C_x{}^{h})=(\nabla_c\nabla_b\xi^{h}+K_{kji}{}^{h}\xi^{k}B_c{}^{j}B_b{}^{i})\varepsilon-(\delta\Gamma_{cb}{}^{a})B_a{}^{h},$$

from which, using (2.11),

$$(\delta h_{cb}{}^{x})C_x{}^{h}+h_{cb}{}^{x}(\eta_x{}^{a}B_a{}^{h}+\eta_x{}^{y}C_y{}^{h})\varepsilon = (\nabla_c\nabla_b\xi^{h}+K_{kji}{}^{h}\xi^{k}B_c{}^{j}B_b{}^{i})\varepsilon-(\delta\Gamma_{cb}{}^{a})B_a{}^{h}.$$

Thus we have

(3.2) $$\delta\Gamma_{cb}{}^{a}=(\nabla_c\nabla_b\xi^{h}+K_{kji}{}^{h}\xi^{k}B_c{}^{j}{}_{b}{}^{i})B^{a}{}_{h}\varepsilon-h_{cb}{}^{x}\eta_x{}^{a}\varepsilon,$$
$$\delta h_{cb}{}^{x}=-h_{cb}{}^{y}\eta_y{}^{x}\varepsilon+(\nabla_c\nabla_b\xi i+K_{kji}{}^{h}\xi^{k}B_c{}^{j}B_b{}^{i})C^{x}{}_{h}\varepsilon,$$

from which, using (1.12) and (2.8),

(3.3) $$\delta h_{cb}{}^{x}=[\xi^{d}\nabla_d h_{cb}{}^{x}+h_{eb}{}^{x}(\nabla_c\xi^{e})+h_{ce}{}^{x}(\nabla_b\xi^{e}) -h_{cb}{}^{y}\eta_y{}^{x}]\varepsilon+[\nabla_c\nabla_b\xi^{x}+K_{kji}{}^{h}C_y{}^{k}B_c{}^{j}{}_{b}{}^{i}C^{x}{}_{h}\xi^{y} -h_{ce}{}^{x}h_b{}^{e}{}_{y}\xi^{y}]\varepsilon.$$

Substituting (2.8) and (2.13) into (3.2) and using equations (1.13) of Gauss and those (1.14) of Codazzi, we have

(3.4) $$\delta\Gamma_{cb}{}^{a}=(\nabla_c\nabla_b\xi^{a}+K_{dcb}{}^{a}\xi^{d})\varepsilon -[\nabla_c(h_b{}^{a}{}_{x}\xi^{x}]+\nabla_b(h_c{}^{a}{}_{x}\xi^{x})-\nabla^{a}(h_{cbx}\xi^{x})]\varepsilon.$$

A variation of a submanifold for which $\delta\Gamma_{cb}{}^{a}=0$ is said to be *affine.*

In this section we suppose that a variation of the submanifdol M^n is isometric, we have from (2.18)

(3.5) $$\nabla_c\xi_b+\nabla_b\xi_c=2h_{cbx}\xi^{x},$$

from which,

(3.6) $$(\nabla_c\xi_b)(h^{cb}{}_{y}\xi^{y})=(h_{cbx}\xi^{x})(h^{cb}{}_{y}\xi^{y}),$$

(3.7) $$\nabla_b \xi^b = h_b{}^b{}_x \xi^x.$$

Since an isometric is affine, we have from (3.4)

$$\nabla^c \nabla_c \xi_a + K_{ca} \xi^c - 2\nabla^c (h_{cax} \xi^x) + \nabla_a (h_b{}^b{}_x \xi^x) = 0,$$

or, transvecting ξ^a

$$(\nabla^c \nabla_c \xi_a) \xi^a + K_{cb} \xi^c \xi^b - 2\xi^b \nabla^c (h_{cbx} \xi^x) + \xi^c \nabla_c (h_b{}^b{}_x \xi^x) = 0.$$

If we take account of (3.7), then the equation above can be written as

(3.8) $$(\nabla^c \nabla_c \xi_a) \xi^a + K_{cb} \xi^c \xi^b + 2 (h_{cbx} \xi^x)(h^{cb}{}_y \xi^y) - (h_b{}^b{}_x \xi^x)^2 - 2\nabla^c (h_{cbx} \xi^x \xi^b) + \nabla^c (h^b{}_{bx} \xi^x \xi_c) = 0.$$

On the other hand, we have

(3.9) $$(1/2) \Delta (\xi_a \xi^a) = (\nabla^c \nabla_c \xi_a) \xi^a + (\nabla_c \xi_b)(\nabla^c \xi^b),$$

where $\Delta = g^{cb} \nabla_c \nabla_b$.

Substituting (3.8) into (3.9) and using (3.6), we find

(3.10) $$\nabla^c W_c + K_{cb} \xi^c \xi^b + (h_{cbx} \xi^x)(h^{cb}{}_y \xi^y) - (h_b{}^b{}_x \xi^x)^2 - (\nabla_c \xi_b - h_{cbx} \xi^x)(\nabla^c \xi^b - h^{cb}{}_y \xi^y) = 0,$$

where we have put

$$W_c = (1/2) \nabla_c (\xi_a \xi^a) - 2h_{cbx} \xi^x \xi^b + (h_b{}^b{}_x \xi^x) \xi_c.$$

Applying Green's theorem to (3.10), we see that

$$\int_M [K_{cb} \xi^c \xi^b + (h_{cbx} \xi^x)(h^{cb}{}_y \xi^y) - (h_b{}^b{}_x \xi^x)^2 - (\nabla^b \xi_b - h_{cbx} \xi^x)(\nabla^c \xi^b - h^{cb}{}_y \xi^y)] dV = 0.$$

Taking account of (3.7), we have

PROPOSITION 3.1. *If an infinitesimal variation $\bar{x}^h = x^h + (\xi^a B_a{}^h + \xi^x C_x{}^h)\varepsilon$ of a compact orientable submanifold is isometric and satisfies*

$$K_{cb} \xi^c \xi^b + (h_{cbx} \xi^x)(h^{cb}{}_y \xi^y) \leqq 0,$$

then ξ^b is a harmonic vector field.

We now compute the variation of the mean curvature vector.

For a variation of the submanifold, we have

$$\delta (g^{cb} h_{cb}{}^x) = (\delta g^{cb}) h_{cb}{}^x + g^{cb} \delta h_{cb}{}^x.$$

Substituting (2.19) and (3.3) into this, we find

(3.11) $$\delta (g^{cb} h_{cb}{}^x) = [\nabla^c \nabla_c \xi^x + K_{kji}{}^h C_y{}^k B^{ji} C^x{}_h \xi^y$$

$$+(h_{cb}{}^{x})(h^{cb}{}_{y}\xi^{y})+\xi^{c}\nabla_{c}h_{b}{}^{bx}-h_{a}{}^{ay}\eta_{y}{}^{x}]\varepsilon,$$

where $B^{ji}=B_{c}{}^{j}{}_{b}{}^{i}g^{cb}$.

If a variation of the submanifold preserves the mean curvature vector, then we have from (3.11)

$$(\nabla^{c}\nabla_{c}\xi_{x})\xi^{x}=-K_{kjih}C_{y}{}^{k}B^{ji}C_{x}{}^{h}\xi^{y}\xi^{x}-(h_{cbx}\xi^{x})(h^{cb}{}_{y}\xi^{y})$$
$$-(\nabla_{c}h_{b}{}^{b}{}_{x})\xi^{x}\xi^{c}+\eta_{yx}\xi^{x}h_{b}{}^{by}. \tag{3.12}$$

We have the equation of Gauss (1.13)

$$K_{cb}\xi^{c}\xi^{b}=K_{kjih}(\xi^{c}B_{c}{}^{k})B^{ji}(\xi^{b}B_{b}{}^{h})+h_{e}{}^{e}{}_{x}(h_{cb}{}^{x}\xi^{c}\xi^{b})$$
$$-(h_{ce}{}^{x}\xi^{c})(h_{b}{}^{e}{}_{x}\xi^{b}). \tag{3.13}$$

On the other hand, we have

$$(1/2)\Delta(\xi_{a}\xi^{a}+\xi_{x}\xi^{x})$$
$$=(\nabla^{c}\nabla_{c}\xi_{a})\xi^{a}+(\nabla^{c}\nabla_{c}\xi_{x})\xi^{x}+(\nabla_{c}\xi_{b})(\nabla^{c}\xi^{b})+(\nabla_{c}\xi_{x})(\nabla^{c}\xi^{x}). \tag{3.14}$$

Substituting (3.8) and (3.12) into (3.14) and taking account of (2.7), (3.6) and (3.13), we have

$$\nabla^{c}u_{c}=-K_{kjih}\xi^{k}B^{ji}\xi^{h}-2(h_{cbx}\xi^{x})(h^{cb}{}_{y}\xi^{y})$$
$$+(h_{b}{}^{b}{}_{x}\xi^{x})^{2}+(h_{ce}{}^{x}\xi^{c})(h_{b}{}^{e}{}_{x}\xi^{b})+(\nabla_{c}\xi_{x})(\nabla^{c}\xi^{x})$$
$$+(\nabla_{c}\xi_{b}-h_{cbx}\xi^{x})(\nabla^{c}\xi^{b}-h^{cb}{}_{y}\xi^{y})$$
$$-(\nabla_{c}h_{b}{}^{b}{}_{x})\xi^{x}\xi^{c}+\eta_{yx}\xi^{x}h_{b}{}^{by}-(h_{e}{}^{e}{}_{x})(h_{cb}{}^{x}\xi^{c}\eta^{b}), \tag{3.15}$$

where we have put

$$u_{c}=(1/2)\nabla_{c}(\xi_{a}\xi^{a}+\xi_{x}\xi^{x})-2h_{cbx}\xi^{x}\xi^{b}+(h_{b}{}^{b}{}_{x}\xi^{x})\xi^{c}.$$

A variation carries a minimal submanifold into a minimal submanifold, that is, $\delta(g^{cb}h_{cb}{}^{x})=0$ and $h_{b}{}^{b}{}_{x}=0$, then we say that the variation is *minimal*.

For a minimal variation, we have from (3.15)

$$\nabla^{c}u_{c}=-K_{kjih}\xi^{k}B^{ji}\xi^{h}-2(h_{cbx}\xi^{x})(h^{cb}{}_{y}\xi^{y})$$
$$+(h_{ce}{}^{x}\xi^{c})(h_{b}{}^{e}{}_{x}\xi^{b})+(\nabla_{c}\xi_{x})(\nabla^{c}\xi^{x})$$
$$+(\nabla_{c}\xi_{b}-h_{cbx}\xi^{x})(\nabla^{c}\xi^{b}-h^{cb}{}_{y}\xi^{y}) \tag{3.16}$$

Thus, we have the following:

LEMMA 3.2. *Let M^n be a compact orientable submanifold of a Riemannian manifold. If an infinitesimal variation $\bar{x}^h=x^h+\xi^h\varepsilon$ of M^n is isometric and minimal, then we have*

(3.17) $$\int_M [K_{kjih}\xi^k B^{ji}\xi^h + 2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y)] dV \geqq 0.$$

Moreover, if $K_{kjih}\xi^k B^{ji}\xi^h + 2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y) \leqq 0$, *then the variation is parallel.*

The final assertion then follows from $K_{kjih}\xi^k B^{ji}\xi^h + 2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y) = 0$, $\nabla_c\xi_x = 0$ and $h_{ce}{}^x\xi^e = 0$.

§ 4. Anti-invariant minimal variations.

In this section we suppose that an anti-invariant variation $\bar{x}^h = x^h + \xi^h\varepsilon$, where $g_{ji}\xi^j\xi^i > 0$ of the submanifold of a Kaehlerian manifold.

First of all, combining Theorem 2.3 and Lemma 3.2, we have

THEOREM 4.1. *Suppose that an anti-invariant variation of the submanifold* M^n *of a Kaehlerian manifold is minimal and preserves.* $f_b{}^x$. *If* M^n *is compact and orientable, and satisfies*

$$K_{kjih}\xi^k B^{ji}\xi^h + 2(h_{cbx}\xi^x)(h^{cb}{}_y\xi^y) \leqq 0,$$

then the variation is parallel.

We now suppose that the ambient Kaehlerian manifold M^{2m} is of constant holomorphic sectional curvature k. Then we have

(4.1) $$K_{kjih} = (1/4)k[g_{kh}g_{ji} - g_{jh}g_{ki} - F_{kh}F_{ji} - F_{jh}F_{ki} - 2F_{kj}F_{ih}].$$

Suppose also that a submanifold M^n of M^{2m} is anti-invariant. Then we have from (1.5), (1.14) and (2.7)

(4.2) $$K_{kjih}\xi^k B^{ji}\xi^h = k/4\ [(n-1)\xi_a\xi^a + n\xi_x\xi^x + 3(f_b{}^x\xi_x)(f^{by}\xi_y)],$$

(4.3) $$\nabla_d h_{cb}{}^y - \nabla_c h_{db}{}^y = 0.$$

From these facts we prove

THEOREM 4.2. *Suppose that* M^{2m} *is a Kaehlerian manifold of constant holomorphic sectional curvature* $k \leqq 0$ *and that* M^n *is a compact orientable anti-invariant submanifold of* M^{2m}. *If an anti-invariant minimal variation preserves* $f_b{}^x$, *then the variation is parallel and* $k=0$.

Proof. Since M^n is minimal, we see that from $\nabla^c h_{cb}{}^x = 0$. Thus we have

(4.4) $$\nabla^c(h_{cbx}\xi^x\xi^b) = (h_{cbx}\xi^x)(\nabla^c\xi^b) + (h_{cbx}\xi^b)(\nabla^c\xi^x).$$

According to Theorem 2.3, $\delta f_b{}^x = 0$ implies a isometric variation and consequently (3.6) and (3.16) are valid.

If we substitute (4.2) and (4.4) into (3.16) and take account of (3.6), then

$$\nabla^c(u_c + 2h_{cbx}\xi^x\xi^b) = -k/4\ [(n-1)\xi_a\xi^a + n\xi_x\xi^x + 3(f_b{}^x\xi_x)(f^{by}\xi_y)]$$

$$+(\nabla_c\xi_b-h_{cbx}\xi^x)(\nabla^c\xi^b-h^{cb}{}_y\xi^y)$$

$$+(\nabla_c\xi_x+h_{cex}\xi^e)(\nabla^c\xi^x+h^c{}_d{}^x\xi^d).$$

Thus, by Green's theorem, we have $k=0$, $\nabla_c\xi_b=h_{cbx}\xi^x$ and $\nabla_c\xi_x+h_{cbx}\xi^b=0$, which proves the theorem.

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