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ANTI-INVARIANT SUBMANIFOLD WITH TRIVIAL NORMAL CONNECTION IN S^{2n+1} AND CP^n

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Introduction. In their recent paper [4], Yano and Kon proved the following

THEOREM Y-K. *Let M be a compact n-dimensional ($n>1$) anti-invariant submanifold of $\boldsymbol{C}^n$ with trivial normal connection. If the mean curvature vector field is parallel with respect to the normal connection, then M is either totally geodesic or $S^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n)$, where $S^1(r_i)$ denotes the circle of radius r_i.*

The purpose of the paper is to prove analgous theorems for submanifolds of a complex projective space and of a sphere. The main tool for this purpose is the theory of immersion compatible with a submersion, which is developed by Lawson [1] and the present author [2]. In §1, we state briefly the necessary results of the theory for later use. In §2, we try to regard a submanifold of a sphere as a submanifold of $\boldsymbol{C}^{n+1}$ which is an ambient space of the sphere. As easy consequences of these consideration we have analogous theorems to that of Yano and Kon. This will be stated in §3.

§1. Immersion that is compatible with the Hopf-fibration.

Let S^{2n+1} be an odd-dimensional unit sphere in a $(2n+2)$-dimensional Euclidean space which is identified with $(n+1)$-dimensional complex space $\boldsymbol{C}^{n+1}$ and J the natural almost complex structure of $\boldsymbol{C}^{n+1}$. The image $\tilde{V}=\tilde{J}\tilde{N}$ of the outward unit normal vector field $\tilde{N}$ to S^{2n+1} by $\tilde{J}$ defines a unit tangent vector field on S^{2n+1} and the integral curves of $\tilde{V}$ are great circles S^1 in S^{2n+1} which are fibres of the Hopf-fibration : $S^1\rightarrow S^{2n+1}\xrightarrow{\pi} CP^n$. Let M be a submanifold of CP^n of real codimension p and the natural immersion $\pi^{-1}(M)$ into S^{2n+1} is compatible with the Hopf-fibration. In this case we have the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{i} & S^{2n+1} \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{i} & CP^n \end{array}$$

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The immersion i is an isometry on the fibres. The diagram implies that for unit vertical vector field $\bar{V}$ of $\pi^{-1}(M)$, $\tilde{V}=i(\bar{V})$ is also unit vertical vector field of S^{2n+1} and that for any tangent vector X to M, the horizontal lift $i(X)^L$ coincide with $i(X^L)$. The almost complex structure J of CP^n is nothing but the fundamental tensor of the submersion π andthe Riemannian strudture of CP^n is characterized by the fact that π is a Riemannian submersion [3]. Let N_A $(A=1,\cdots,p)$ be mutually orthonormal normal vectors at a point $x\in M$ and extend them to local fields in a neighborhood of x. Then their horizontal lifts N_A^L $(A=1,\cdots,p)$ are also mutually orthonormal normal local vector fields to $\pi^{-1}(M)$ in a neighborhood of $y\in\pi^{-1}(x)$. The transforms $Ji(X)$ and JN_A can be respectively written by

$$Ji(X)=i(FX)+\sum_{A=1}^{p}u_A(X)N_A, \tag{1.1}$$

$$JN_A=-i(U_A)+\sum_{B=1}^{p}\lambda_{AB}{}^{L}B_B, \tag{1.2}$$

and we easily see that $u_A(X)=g(U_A,X)$, where g is the induced Riemannian metric of M from the Fubini-Study metric G of CP^n. The linear transformation F on $T(M)$ thus defined is nothing but the fundamental tensor of the submersion $\pi:\pi^{-1}(M)\to M$. We denote by H_A and $\bar{H}_A$ the corresponding second fundamentl tensors to N_A and N_A^L respectively. Then it is known [2] that

$$\text{trace } \bar{H}_A=(\text{trace} H_A)^L, \tag{1.3}$$

$$\bar{g}(\bar{H}_A\bar{V},\bar{V})=0,\quad \bar{g}(\bar{H}_AX^L,Y^L)=g(H_AX,Y)^L, \tag{1.4}$$

where $\bar{g}$ denotes the Riemannian metric of $\pi^{-1}(M)$.

Let D^N and $\bar{D}^N$ respectively be the connections of the normal bundles which induced from the ambient manifolds CP^n and S^{2n+1}. By definition of the normal connection, we have [2],

$$g(U_A,X)^L=-\bar{g}(\bar{H}_AX^L,\bar{V}), \tag{1.5}$$

$$\bar{D}_{X^L}{}^{N}N_A{}^{L}=(D_X{}^{N}N_A)^L, \tag{1.6}$$

$$\bar{D}_{\bar{V}}{}^{N}N_A{}^{L}=-\sum_{A=1}^{p}\lambda_{AB}{}^{L}N_B{}^{L}. \tag{1.7}$$

Let H and $\bar{H}$ be the mean curvature vector fields of M and $\pi^{-1}(M)$ respectively. Then we have from (1.3) and (1.5),

$$\begin{aligned}(n+1)\bar{D}_{X^L}\bar{H}&=\sum_{A=1}^{p}[X^L(\text{trace}\bar{H}_A)N_A{}^L+(\text{trace}\bar{H}_A)\bar{D}_{X^L}{}^{N}N_A{}^{L}]\\&=\sum_{A=1}^{p}[X(\text{trace}H_A)N_A+(\text{trace}H_A)D_X{}^{N}N_A]^L\\&=n(D_X{}^{N}H)^L,\end{aligned} \tag{1.8}$$

and

$$(n+1)\bar{D}_{\bar{V}}\bar{H}=\sum_{A=1}^{p}[\bar{V}(\text{trace}\bar{H}_A)N_A{}^L+(\text{trace}\bar{H}_A)D_{\bar{V}}{}^{N}N_A{}^{L}], \tag{1.9}$$

$$=\sum_{B,A=1}^{p}[\bar{V}(\mathrm{trace}H_A)^L N_A{}^L-(\mathrm{trace}H_A)^L\lambda_{AB}{}^L N_B{}^L]$$
$$=-[\sum_{A,B=1}^{p}(\mathrm{trace}H_A)\lambda_{AB}N_B]^L.$$

The curvature tensor of CP^n is given by

$$(1.10)\qquad R'(X',Y')Z'=G(Y',Z')X'-G(X',Z')Y'+G(JY',Z')JX'$$
$$-G(JX',Z')JY'-2G(JX',Y')JZ',$$

and so

$$G(R'(i(X),i(Y))N_A,N_B)=g(U_A,Y)g(U_B,X)-g(U_A,X)g(U_B,Y)$$
$$-2g(FX,Y)\lambda_{AB}.$$

On the other hand, S^{2n+1} being a space of constant curvature 1, relations between normal curvature R^N and $\bar{R}^N$ are given by

$$(1.11)\quad \bar{G}(\bar{R}^N(X^L,Y^L)N_A{}^L,N_B{}^L)=G(R^N(X,Y)N_A,N_B)^L+2g(FX,Y)^L\lambda_{AB}{}^L,$$

where $\bar{G}$ is the Riemannian metric of S^{2n+1}. Furthermore we have

$$\bar{G}(\bar{R}^N(\bar{V},Y^L)N_A{}^L,N_B{}^L)=\bar{g}([\bar{H}_A,\bar{H}_B]\bar{V},Y^L)$$
$$=(\bar{g}(\bar{H}_A\bar{H}_B-\bar{H}_B\bar{H}_A)\bar{V},Y^L)$$
$$=\bar{g}(\bar{H}_B\bar{V},(H_AY)^L)+\bar{g}(\bar{H}_AX^L,\bar{V})\bar{g}(\bar{H}_B\bar{V},\bar{V})-\bar{g}(\bar{H}_A\bar{V},(H_BY)^L)$$
$$-\bar{g}(\bar{H}_BX^L,\bar{V})\bar{g}(\bar{H}_A\bar{V},\bar{V})$$
$$=\bar{g}(\bar{H}_B\bar{V},(H_AY)^L)-\bar{g}(\bar{H}_A\bar{V},(H_BY)^L)$$
$$=-g(U_B,H_AY)^L+g(U_A,H_BY)^L,$$

because of (1.4) and (1.5). Thus we have

$$(1.12)\qquad \bar{G}(\bar{R}^N(\bar{V},Y^L)N_A{}^L,N_B{}^L)=g(H_BU_A-H_AU_B,Y)^L.$$

§2. Anti-invariant submanifolds of S^{2n+1} in C^{n+1}.

Let $\tilde{Q}$ be a linear transformation of the tangent bundle $T(\tilde{M})$ of a differentiable manifold $\tilde{M}$ and M a submanifold of $\tilde{M}$. If the tangent bundle $T(M)$ of M satisfies $\tilde{Q}T(M)\subset T(M)$, we say that the submenifold M is an invariant submanifold of $\tilde{M}$ under the action of $\tilde{Q}$. On the other hand if $\tilde{Q}$ $T(M)$ is orthogonal to $T(M)$, we say that M is an anti-invariant submanifold of $\tilde{M}$ under the action of $\tilde{Q}$. Now we regard the sphere of radius 1 as a Sasakian manifold with natural Sasakian structure $(J^L,V,\bar{G})$, where $\bar{G}$ is the induced Riemannian metric of S^{2n+1} from the ambient space $C^{n+1}=E^{2n+2}$. We denote by $\tilde{N}$ the position vector field which defines S^{2n+1} in C^{n+1}. $\tilde{N}$ is also a unit normal vector field to S^{2n+1}. Since J^L is defined from the almost complex structure $\tilde{J}$ of the ambient space C^{n+1} in such a way that

$$Ji_2(\tilde{X})=i_2J^L(\tilde{X})+\bar{G}(\tilde{V},\tilde{X})\tilde{N},$$

where i_2 is the immersion $S^{2n+1}\to C^{n+1}$, $\tilde{X}\in T(S^{2n+1})$, if $\bar{M}$ is an anti-invariant submanifold under the action of J^L, $\bar{M}$ is also anti-invariant under the action of $\tilde{J}$ of C^{n+1}.

Let $\bar{M}$ is an anti-invariant submanifold of codimension n. Denoting by i_1 and $\bar{N}_A$ $(A=1,\cdots,n)$ the immersion $\bar{M}\to S^{2n+1}$ and mutually orthonormal normal vectors to $\bar{M}$ in S^{2n+1} respectively, we have the following Gauss and Weingarten equations for $\bar{X}$, $\bar{Y}\in T(\bar{M})$.

(2.1) $$\bar{\bar{\nabla}}_{i_2i_1(\bar{X})}i_2i_1(\bar{Y})=i_2i_1\bar{\nabla}_{\bar{X}}\bar{Y}+\sum_{A=1}^{n}\bar{g}(\bar{H}_A\bar{X},\bar{Y})i_2N_A-\bar{g}(\bar{X},\bar{Y})N,$$

(2.2) $$\bar{\bar{\nabla}}_{i_1i_2(\bar{X})}i_2(\bar{N}_A)=-i_2i_1(\bar{H}_A\bar{X})+\bar{\bar{D}}_{i_1(\bar{X})}{}^{N}i_2(\bar{N}_A),$$

(2.3) $$\bar{\bar{\nabla}}_{i_2i_1(\bar{X})}\tilde{N}=i_2i_1(\bar{X})+\sum_{A=1}^{n}\bar{\bar{L}}_A(\bar{X})i_1\bar{N}_A,$$

where $\bar{\bar{\nabla}}$, $\bar{\bar{L}}_A$, respectively denote the covariant differentiation with respect to the Euclidean metric of $C^{n+1}$and the third fundamental tensor with respect to the immersion i_2i_1. On the other hand we have

(2.4) $$\begin{aligned}\bar{\bar{\nabla}}_{i_2(i_1(\bar{X}))}i_2(N_A)&=i_2\bar{D}_{i_1(\bar{X})}\ \bar{N}_A+\bar{G}(\bar{\bar{H}}_{n+1}i_1(\bar{X}),\bar{N}_A)\tilde{N}\\&=-i_2i_1\bar{H}_A\bar{X}+i_2\bar{D}_{\bar{X}}{}^{N}\bar{N}_A-\bar{G}(i_1(\bar{X}),\bar{N}_A)\tilde{N}\\&=-i_2i_1\bar{H}_A\bar{X}+i_2\bar{D}_{\bar{X}}{}^{N}\bar{N}_A,\end{aligned}$$

(2.5) $$\bar{\bar{\nabla}}_{i_2(i_1(\bar{X}))}\tilde{N}=-i_2\bar{\bar{H}}_{n+1}i_1(\bar{X})=i_2i_1(\bar{X}).$$

Comparing (2.2), (2.3), (2.4) and (2.5), we have

(2.6) $$\bar{\bar{D}}_{i_1(\bar{X})}i_2(\bar{N}_A)=i_2\bar{D}_{\bar{X}}{}^{N}\bar{N}_A,$$

(2.7) $$\bar{\bar{L}}_A(\bar{X})=0,$$

from which

(2.8) $$\bar{\bar{D}}_{i_1(\bar{X})}{}^{N}\tilde{X}=0.$$

The mean curvature vector field $\bar{\bar{H}}$ of $\bar{M}$ in C^{n+1} is

(2.9) $$\bar{\bar{H}}=[\textstyle\sum_{A=1}^{n}(\mathrm{trace}\bar{H}_A)i_2\bar{N}_A+\mathrm{trace}\ (-I)\tilde{N}]/n+1=n(i_2\bar{H}-\tilde{N})/n+1.$$

Thus we have

(2.10) $$\begin{aligned}\bar{\bar{D}}_{i_2i_1(\bar{X})}{}^{N}\bar{\bar{H}}&=n(\bar{\bar{D}}_{i_1(\bar{X})}i_2{}^{N}\bar{H}-\bar{\bar{D}}_{i_1(\bar{X})}\tilde{N})/n+1\\&=n\textstyle\sum_{A=1}^{n}[\bar{\bar{D}}_{i_1(\bar{X})}{}^{N}(\mathrm{trace}\ \bar{H}_A)i_2\bar{N}_A]/n+1\\&=n\textstyle\sum_{A=1}^{n}[\bar{X}(\mathrm{trace}\ \bar{H}_A)i_2\bar{N}_A+i_2\bar{D}_{\bar{X}}{}^{N}\bar{N}_A]/n+1\\&=n(i_2\bar{D}_{\bar{X}}{}^{N}\bar{H}),\end{aligned}$$

which shows that if the mean curvature vector field $\bar{H}$ of $\bar{M}$ is parallel with respect to the normal connection in S^{2n+1}, so is the mean curvature vector field $\bar{\bar{H}}$ of $\bar{M}$ in C^{n+1}.

Let $\bar{\bar{R}}^N(\bar{X},\bar{Y})$ be the normal curvature of $\bar{M}$ in C^{n+1}. Then from (2.6) it follows that

$$\bar{\bar{R}}^N(\bar{X},\bar{Y})i_2\bar{N}_A=(\bar{\bar{D}}_{\bar{Y}}{}^N\bar{\bar{D}}_{\bar{X}}{}^N-\bar{\bar{D}}_{\bar{X}}{}^N\bar{\bar{D}}_{\bar{Y}}{}^N-\bar{\bar{D}}_{[\bar{X},\bar{Y}]})i_2\bar{N}_A$$
$$=i_2(\bar{D}_{\bar{Y}}{}^N\bar{D}_{\bar{X}}{}^N-\bar{D}_{\bar{X}}{}^N{}_{\bar{Y}}\bar{D}^N-\bar{D}_{[\bar{X},\bar{Y}]}{}^N)\bar{N}_A=i_2\bar{R}^N(\bar{X},\bar{Y})\bar{N}_A.$$

This, together with (2.7), implies that $\bar{\bar{R}}^N(\bar{X},\bar{Y})=0$. We have proved the

LEMMA 2.1. *Let $\bar{M}$ be a submanifold of S^{2n+1} with trivial normal connection. If the mean curvature vector field of $\bar{M}$ in S^{2n+1} is parallel with respect to the normal connection, then, as a submanifold of C^{n+1}, $\bar{M}$ is of trivial normal connection and its mean curuature vector field in C^{n+1} is also parallel with respect to the normal connection induced from C^{n+1}.*

Combining Lemma 2.1 and Theorem Y-K, we have

THEOREM 2.2. *Let $\bar{M}$ be a compact $n+1$ dimensional anti-invariant submanifold of a unit sphere S^{2n+1} under the action of the natural Sasakian structure J^L and the vector field $\tilde{V}$ is always tangent to $\bar{M}$. If $\bar{M}$ is of trivial normal connection and the mean curvature vector field is parallel with respect to the normal connection, then $\bar{M}$ is $S^1(r_1)\times\cdots\times S^1(r_{n+1})$.*

§3. n-dimensional anti-invariant submanifolds of CP^n.

In this section we consider such an n-dimensional submanifold M of a complex projective space CP^n that at any point of the submanifold the tangent space is anti-invariant under the action of the almost complex structure J of CP^n. Since the submanifold is n-dimensional we have from (1.1) and (1.2),

(3.1) $$Ji(X)=\sum_{A=1}^{n}u_A(X)N_A,$$

(3.2) $$JN_A=-i(U_A),$$

that is, $F=0$, $\lambda_{AB}=0$. Hence we get

(3.3) $$\bar{D}_{\bar{Y}}\bar{H}=0,$$

(3.4) $$\bar{G}(\bar{R}^N(X^L,Y^L)N_A{}^L,\ N_B{}^L)=G(R^N(X,Y)N_A,N_B)^L,$$

because of (1.9) and (1.11).

If the mean curvature vector field of the anti-invariant submanifold is parallel with respect to the induced connection of the normal bundle, (1.8) and (3.3) imply that so is the mean curvature vector field $\bar{H}$ of $\pi^{-1}(M)$. Furthermore, if M is of trivial connection and satisfies

(3.5) $$H_BU_A=H_AU_B,$$

for $A, B=1, 2, \cdots, n$, (1.12) and (3.4) show that $\pi^{-1}(M)$ is of trivial normal connection. On the other hand, by construction of the almost complex structure of CP^n, $(J^L, \tilde{V})$ defines the natural contact metric structure of S^{2n+1}. Thus $\pi^{-1}(M)$ is anti-invariant under the action of J^L if and only if M is anti-invariant under J. Hence we have

THEOREM 3.1. *Let M be a compact n-dimensional $(n>1)$ anti-invariant submanifold of a complex projective space CP^n with trivial normal connection. If the mean curvature vector field of M is parallel uith respect to the normal connection and satisfies* (3.5), *then $\pi^{-1}(M)$ is $S^1(r_1)\times\cdots\times S^1(r_{n+1})$, where $S^1(r_i)$ denotes the circle of radius r_i. Consequently M is diffeomorphic to n-product of circles.*

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