J. Korean Math. Soc.
Vol.14, No.1, 1977.

ON THE RICCI TENSORS OF PARTICULAR FINSLER SPACES

BY HIDEO SHIMADA

In the present paper we shall study various Ricci tensors in particular Finsler spaces. The problem of the Ricci tensors of Finsler spaces is stated, for example, in the paper [15][1] § 2 and will be important in applications of Finsler geometry to the theoretical physics. One of the difficulties of the problem is that the Ricci tensors defined from the h-curvature tensor R_{hijk} and hv-curvature tensor P_{hijk} are not symmetric in general, contrary to the case of Riemannian geometry. This studies was promoted by Professor Y. Takano's report [25], continuing [15] § 2.

In § 1 two kinds of the hv-Ricci tensors, denoted by $P_{ij}^{(1)}$ and $P_{ij}^{(2)}$, and the h-Ricci tensor R_{ij} are introduced. The purpose of the next section is to consider the Bianchi identities and to produce various identities related to the Ricci tensors. The so-called conservation law is important in the physics. We find a tensor field which satisfies the law under some assumption (Theorem 1). The third section is devoted to studying the Ricci tensors of a C-reducible Finsler space, which is defined by Prof. M. Matsumoto [7] and will be important in the physics. In § 4 we shall touch upon isotropic Finsler spaces due to H. Akbar-Zadeh [1]. We shall treat, in § 5, Finsler spaces of scalar curvature owing to L. Berwald [2]. It is shown that in such a space the h-Ricci tensor R_{ij} is symmetric and the condition $R_{ij}=\nu g_{ij}$ yields "of constant curvature" (Theorem 10). Furthermore we pay attension to Finsler spaces of scalar curvature which satisfy the C-reducibility. The final section is a list of various known results of the v-curvature tensor S_{hijk} and v-Ricci tensor S_{ij}.

The author wishes to express here his sincere gratitude to Prof. Dr. M. Matsumoto, who not only suggested valuable improvement in the present paper, but also whose advices and criticisms during my stay at Kyoto University from Sept. 1976 to Fed. 1977 were of inestimable benefit.

§ 1. The hv-and h-Ricci tensors.

Let F^n be an n-dimensional Finsler space with the fundamental function

Received by the editors Feb. 15, 1977.
This research is supported by the Department of Education of Japan Scholarship grant 1976.

1) Numbers in brackets refer to the references at the end of the paper.

$L(x, y)$, $(y=\dot{x})$. We denote by $g_{ij}(x, y)=(\partial^2L^2/\partial y^i\partial y^j)/2$ the fundamental tensor. The angular metric tensor h_{ij} is given as $h_{ij}=g_{ij}-l_il_j$, where $l_i=\partial L/\partial y^i$ is the normalized element of support. Hereafter the terminologies and notations are the same as in the monograph [12].

The hv-curvature tensor P_{hijk} (cf. (2.3)) satisfies the four identities ([12] §17):

$$P_{hijk}=-P_{ihjk}, \tag{1.1}$$

$$P_{hiok}=P_{hiko}=0, \tag{1.2}$$

$$S_{(hij)}\{P_{hijk}\}=0, \tag{1.3}$$

$$S_{(hik)}\{P_{hijk}\}=0, \tag{1.4}$$

where $S_{(hij)}$ means cyclic permutation of indices h, i, j and summation and the index o means contraction by the element of support y^i.

By virtue of (1.1), (1.3) and (1.4) the number N of the independent components of the hv-curvature tensor P_{hijk} is given in the formula

$$N=n^2(n-1)(n+4)/6.$$

From this we have, for example, $N=4$ $(n=2)$, $N=21$ $(n=3)$, $N=64$ $(n=4)$ and so on.

Next we define the *hv-Ricci tensors* from the hv-curvature tensor P_{hijk} in the following forms:

$$P_{ij}^{(1)}=P_{i\ sj}^{\ s},\quad P_{ij}^{(2)}=P_{i\ js}^{\ s},$$

where $P_{i\ sj}^{\ s}=g^{ms}P_{imsj}$ and $P_{i\ js}^{\ s}=g^{ms}P_{imjs}$.

As a matter of course $P_{ij}^{(1)}\neq P_{ij}^{(2)}$ in general, but we can see that the skew-symmetric parts of $P_{ij}^{(1)}$ and $P_{ij}^{(2)}$ are equal to each other, which was suggested by Prof. M. Matsumoto.

PROPOSITION 1. *The skew-symmetric parts of the two hv-Ricci tensors* $P_{ij}^{(1)}$ *and* $P_{ij}^{(2)}$ *are equal:*

$$P_{ij}^{(1)}-P_{ji}^{(1)}=P_{ij}^{(2)}-P_{ji}^{(2)}=P_{ij\ s}^{\ \ s},$$

where $P_{ij\ s}^{\ \ s}=g^{sm}P_{ijms}$.

Proof. Multiplying (1.3) by g^{mi} and summing over m and k we have

$$P_{hj}^{(2)}-P_{jh}^{(2)}=P_{hj\ m}^{\ \ m}.$$

Similarly from (1.4) we get

$$P_{ik}^{(1)}-P_{ki}^{(1)}=P_{ik\ m}^{\ \ m}.$$

Consequently we have proved the identities as above.

REMARK 1. A Finsler space with $P_{hijk}=P_{hikj}$ is called *P-symmetric* ([13],

[15]). If a Finsler space is P-symmetric, we have the unique hv-Ricci tensor $P_{ij}=P_{ij}^{(1)}=P_{ij}^{(2)}$; this is a very convenient fact, but, contrary to our expectation, the scalar curvature $P=P_{ij}g^{ij}$ necessarily vanishes in this case ([15] § 2).

REMARK 2. A Finsler space with the Cartan connecton $C\Gamma$ is P-symmetric, if and only if the v-curvature tensor S_{hijk} satisfies the equation ([12] (17.25), [15] Prop. 2)

$$S_{hijk|o}=0$$

where ($|$) denotes the h-covariant differentiation with respect to $C\Gamma$.

We turn the consideration to the h-curvature tensor $R_h{}^i{}_{jk}$ which is given in the form

$$R_h{}^i{}_{jk}=A_{(jk)}\{\delta_k F_h{}^i{}_j+F_h{}^r{}_j F_r{}^i{}_k\}+C_h{}^i{}_r R^r{}_{jk}, \tag{1.5}$$

where $A_{(jk)}$ means interchange of indices j, k and subtraction and $\delta_k=\partial_k-N^j{}_k\dot{\partial}_j$.

If we adopt the Cartan connection $C\Gamma$ as the Finsler connection, the following identities hold ([12] (17.9), (17.10). (22.7), (22.8)):

$$R_{hijk}=-R_{ihjk},\quad R_{hijk}=-R_{hikj}, \tag{1.6}$$

$$S_{(ijk)}\{R_{hijk}\}=-S_{(ijk)}\{C_h{}^r{}_i R_{rjk}\}, \tag{1.7}$$

$$R_{jkhi}=R_{hijk}+N_{hijk}, \tag{1.8}$$

where $N_{hijk}=A_{(jk)}\{C_i{}^r{}_j R_{rhk}-C_h{}^r{}_j R_{rik}\}$.

REMARK 3. In a Finsler space of scalar curvature, the h-curvature tensor R_{hijk} satisfies $S_{(ijk)}\{R_{hijk}\}=0$ and $R_{jkhi}=R_{hijk}$ as Riemannian curvature tensor. See § 5.

We define the *h-Ricci tensor* R_{ij} from the h-curvature tensor R_{hijk} in the form

$$R_{ij}=R_i{}^s{}_{js}.$$

The h-Ricci tensor R_{ij} is also equal to $R^s{}_{isj}$ because of $R_{hijk}=R_{ihkj}$.

In case of the Cartan connection $C\Gamma$ it is observed that

PROPOSITION 2. *The skew-symmetric part of the h-Ricci tensor R_{kh} is given by the equation*

$$R_{kh}-R_{hk}=C_i R^i{}_{kh}+C_h{}^i{}_r R^r{}_{ik}-C_k{}^i{}_r R^r{}_{ih},\quad (C_i=C_i{}^m{}_m). \tag{1.9}$$

Proof. From (1.5) we obtain easily

$$R_{kh}-R_{hk}=\delta_k F_h{}^i{}_i-\delta_h F_k{}^i{}_i+C_h{}^i{}_r R^r{}_{ik}-C_k{}^i{}_r R^r{}_{ih},$$

where $F_h{}^i{}_i = \gamma_h{}^i{}_i - C_i N^i{}_h = \delta_h \log\sqrt{g}$ ([12] (17.3)′). Applying δ_k to the above, we get

$$\delta_k F_h{}^i{}_i - \delta_h F_k{}^i{}_i = C_i R^i{}_{kh}.$$

Thus the proof is completed.

It is noted that the contraction of (1.7) by g^{ij} yields immediately another proof of (1.9).

§2. The Bianchi identities and Ricci tensors.

In the theory of general Finsler connections, devoted in the monograph [12], we have four Jacobi identites in combination with two vector fields, called the h-and v-basic vector fields. From each Jacobi identity, we obtain three identities, which show the vanishing of the h-horizontal part, v-horizontal part and vertical part of the Jacobi identity. Hence there are twelve identities. Because one of these identities is trivial, we have finally eleven Bianchi identities which are classified into four groups ([12] §11).

We are specially concerned with the Cartan connection $C\Gamma$. The four groups of the Bianchi identities of $C\Gamma$ are as follows ([12] §17):

The first group

(BC-I-1) $$S_{(ijk)}\{C_i{}^h{}_r R^r{}_{jk} - R_i{}^h{}_{jk}\} = 0,$$

(BC-I-2) $$S_{(ijk)}\{P^h{}_{ir} R^r{}_{jk} + R^h{}_{ij|k}\} = 0,$$

(BC-I-3) $$S_{(ijk)}\{P_m{}^h{}_{ir} R^r{}_{jk} + R_m{}^h{}_{ij|k}\} = 0.$$

The second group

(BC-II-1) $$A_{(ij)}\{C_j{}^h{}_{k|i} + C_i{}^h{}_r P^r{}_{jk} - P_i{}^h{}_{jk}\} = 0,$$

(BC-II-2) $$R^h{}_{ij}|_k - R_k{}^h{}_{ij} + A_{(ij)}\{R^h{}_{ir} C_j{}^r{}_k + P^h{}_{ir} P^r{}_{jk} + P^h{}_{jk|i}\} = 0,$$

(BC-II-3) $$R_m{}^h{}_{ij}|_k + S_m{}^h{}_{kr} R^r{}_{ij} + A_{(ij)}\{R_m{}^h{}_{ir} C_j{}^r{}_k + P_m{}^h{}_{ir} P^r{}_{jk} + P_m{}^h{}_{jk|i}\} = 0,$$

where (|) denotes the v-covariant differentiation with respect to $C\Gamma$.

The third group

(BC-III-1) $$A_{(jk)}\{C_i{}^h{}_j|_k - C_r{}^h{}_k C_j{}^r{}_i\} - S_i{}^h{}_{jk} = 0,$$

(BC-III-2) $$A_{(ij)}\{P^h{}_{ri} C_k{}^r{}_j - P^h{}_{kj}|_i + P_i{}^h{}_{kj}\} = 0,$$

(BC-III-3) $$S_m{}^h{}_{ij|k} + A_{(ij)}\{P_m{}^h{}_{ri} C_k{}^r{}_j - S_m{}^h{}_{ir} P^r{}_{kj} - P_m{}^h{}_{kj}|_i\} = 0.$$

The fourth group

(BC-IV-1) $$S_{(ijk)}\ \{S_i{}^h{}_{jk}\} = 0,$$

(BC-IV-2) $$S_{(ijk)}\ \{S_m{}^h{}_{ij}|_k\} = 0.$$

It is remarked that (BC-I-2), (BC-II-2) and (BC-III-2) is a consequence of (BC-I-3), (BC-II-3) and (BC-III-3) respectively.

From (BC-I-1), contracting for h and j we have (1.9) because (BC-I-1) is quite the same with (1.7).

The contraction of (1.9) by y^h yields

$$R_{ko}-R_{ok}=C_iR^i{}_{ko}+C_k{}^i{}_rR^r{}_{oi}. \tag{2.1}$$

By virtue of the metrical property of the Cartan connection $C\Gamma$, a contraction and h-(or v-) covariant differentiation are commutative.

Next we contract (BC-I-3) for h and i to obtain

$$R_{mk|i}-R_{mi|k}-P_{mr}{}^{(1)}R^r{}_{ki}=R_m{}^h{}_{ki|h}+P_m{}^h{}_{ir}R^r{}_{hk}-P_m{}^h{}_{kr}R^r{}_{hi}. \tag{2.2}$$

In Riemannian geometry (2.2) yields the important equation $R_{|i}=2R^r{}_{i|r}$ by contracting by g^{mk} ([22] p. 18).

Contracting (2.2) by y^m and y^my^k, we have respectively

$$R_{ok|i}-R_{oi|k}-P_{or}{}^{(1)}R^r{}_{ki}=R^r{}_{ki|r}+P^s{}_{ir}R^r{}_{sk}-P^s{}_{kr}R^r{}_{si}, \tag{2.2)'}$$

$$R_{oo|i}-R_{oi|o}-P_{or}{}^{(1)}R^r{}_{oi}=R^r{}_{oi|r}+P^s{}_{ir}R^r{}_{so}. \tag{2.2)''}$$

Next the Bianchi identity (BC-II-1) is, by virtue of (1.1) and (1.3), rewritten as

$$P_{hijk}=C_{ijk|h}-C_{hjk|i}+P_{ikr}C_j{}^r{}_h-P_{hkr}C_j{}^r{}_i, \tag{2.3}$$

which is nothing but the well-known representation of the hv-curvature tensor P_{hijk} of the Cartan connection $C\Gamma$ ([12] (17.23)).

From (2.3) the hv-Ricci tensors $P_{hk}{}^{(1)}$ and $P_{hk}{}^{(2)}$ are written as follows:

$$\begin{aligned} P_{hk}{}^{(1)}&=C_{k|h}-C_h{}^s{}_{k|s}+P_k{}^s{}_rC_s{}^r{}_h-P_h{}^r{}_kC_r, \quad (P_k{}^s{}_r=g^{sm}P_{kmr}).\\ P_{hk}{}^{(2)}&=C_{k|h}-C_h{}^s{}_{k|s}+C_k{}^r{}_hC_{r|o}-P_h{}^s{}_rC_k{}^r{}_s. \end{aligned} \tag{2.4}$$

Consequently it is easily observed that

PROPOSITION 3. *The hv-Ricci tensors $P_{hk}{}^{(1)}$ and $P_{hk}{}^{(2)}$ of the Cartan connection $C\Gamma$ satisfy the following equations*:

(1) $$P_{ok}{}^{(1)}=P_{ok}{}^{(2)}=C_{k|o},\quad P_{ho}{}^{(1)}=P_{ho}{}^{(2)}=0,$$

(2) $$P_{hk}{}^{(1)}-P_{kh}{}^{(1)}=P_{hk}{}^{(2)}-P_{kh}{}^{(2)}=A_{(hk)}\{C_{k|h}+P_k{}^s{}_rC_s{}^r{}_h\},$$

(3) $$P_{hk}{}^{(1)}-P_{hk}{}^{(2)}=P_k{}^s{}_rC_s{}^r{}_h+P_h{}^s{}_rC_s{}^r{}_k-P_h{}^r{}_kC_r-C_h{}^r{}_kC_{r|o}.$$

REMARK 4. According to Remark 1 the P-symmetry implies that the right hand side of (3) in Proposition 3 vanishes but the inverse does not hold. Further the hv-Ricci tensor P_{ij} is not symmetric even if P-symmetry holds good.

From (BC-II-3) we take the contraction for h and j to get

$$(2.5)\qquad R_{lk}|_i+R_{lr}C_k{}^r{}_i+P_{li}{}^{(1)}{}_{|k}-P_{lr}{}^{(1)}P^r{}_{ki}$$
$$=P_l{}^r{}_{ki|r}-R_l{}^s{}_{kr}C_s{}^r{}_i-P_l{}^s{}_{kr}P^r{}_{si}+S_l{}^s{}_{ir}R^r{}_{sk}.$$

Contracting (2.5) by y^l and y^ly^k, in view of $P_{oi}{}^{(1)}=C_{i|o}$ we get respectively

$$(2.5)'\qquad R_{ok}|_i-R_{ik}+R_{or}C_k{}^r{}_i+C_{i|o|k}-P^r{}_{ki}C_{r|o}$$
$$=P^r{}_{ki|r}-R^s{}_{kr}C_s{}^r{}_i-P^s{}_{kr}P^r{}_{si},$$

$$(2.5)''\qquad R_{oo}|_i+C_{i|o|o}=R_{oi}+R_{io}-R^s{}_{or}C_s{}^r{}_i$$
$$=2R_{oi}+C_rR^r{}_{ic}.$$

In the last equation we refered to (2.1).

In (BC-II-3), we take another contraction for h and i to get

$$(2.6)\qquad P_{lj}{}^{(2)}{}_{|k}-P_{lk}{}^{(2)}{}_{|j}+S_{lr}R^r{}_{jk}=R_l{}^r{}_{jk}|_r+A_{(jk)}\{R_l{}^s{}_{jr}C_k{}^r{}_s+P_l{}^s{}_{jr}P^r{}_{ks}\},$$

where S_{lr} is the *v-Ricci tensor* $S_{lr}=S_l{}^m{}_{rm}$.
Contract (2.6) by y^l and y^ly^k. Paying attention to $y^i{}_{|j}=0$ and $y^i|_j=\delta^i{}_j$, in virtue of (1) of Prop. 3 we get

$$(2.6)'\qquad C_{j|o|k}-C_{k|o|j}=R^r{}_{jk}|_r+R^s{}_{jr}C_k{}^s{}_r-R^s{}_{kr}C_j{}^r{}_s,$$

$$(2.6)''\qquad C_{j|o|o}=R^r{}_{jo}|_r-R_{oj}-R^s{}_{or}C_j{}^r{}_s.$$

We compare (2.6)″ with (2.5)″ : Eliminating the term $C_{i|o|o}$ from them we obtain

$$(2.7)\qquad R_{oo}|_i+R^h{}_{io}|_h-2R_{oi}-R_{io}=0.$$

In Riemannian geometry (2.7) reduces to a trivial equation.

Next from (BC-III-2), in virtue of (1.1) and (1.4) another expression of the hv-curvature tensor P_{hijk} is obtained ([12](17.27)):

$$(2.8)\qquad P_{hijk}=P_{ijk}|_h-P_{hjk}|_i+P_{ikr}C_j{}^r{}_h-P_{hkr}C_j{}^r{}_i.$$

The contraction of (2.8) by g^{ij} and g^{ik} yields respectively

$$(2.4)'\qquad \begin{aligned} P_{hk}{}^{(1)}&=C_{k|o}|_h-P_h{}^s{}_k|_s+P_k{}^s{}_rC_s{}^r{}_h-P_k{}^r{}_kC_r,\\ P_{hk}{}^{(2)}&=C_{k|o}|_h-P_h{}^s{}_k|_s+C_k{}^r{}_hC_{r|o}-P_h{}^s{}_rC_k{}^s{}_r.\end{aligned}$$

This is another representation of (2.4). We make a comparison with (2.4)′ and (2.4) to get the following interesting equation:

$$(2.9)\qquad C_{i|o}|_j-C_{i|j}=P_i{}^r{}_j|_r-C_i{}^r{}_{j|r}.$$

Hence we conclude

PROPOSITION 4. *In a Finsler space F^n with the Cartan connection, the tensor*

$$C_{i|o}|_j - C_{i|j}$$

is written as (2.9) and symmetric in i and j.

From (2.4)′ it is observed that the equation (2) of Proposition 3 is written in the form

$$P_{hk}{}^{(1)} - P_{kh}{}^{(1)} = P_{hk}{}^{(2)} - P_{kh}{}^{(2)} = A_{(hk)}\{C_{k|o}|_h + P_k{}^s{}_r C_s{}^r{}_h\}.$$

Now we are concerned with (BC-III-3). The contraction for h and k yields

(2.10) $$P_{mj}{}^{(1)}|_i - P_{mi}{}^{(1)}|_j = S_m{}^r{}_{ij|r} + A_{(ij)}\{P_m{}^s{}_{ri} C_s{}^r{}_j + S_m{}^s{}_{ri} P^r{}_{sj}\},$$

which corresponds to the equation (2.6). The contraction of (2.10) by y^m gives

(2.10)′ $$C_{j|o}|_i - C_{i|o}|_j = P_{ij}{}^{(1)} - P_{ji}{}^{(1)} + P^s{}_{ri} C_s{}^r{}_j - P^s{}_{rj} C_s{}^r{}_i.$$

Substituting from (2) of Prop. 3 into the above, we get an interesting equation

(2.10)″ $$C_{j|o}|_i - C_{i|o}|_j = C_{j|i} - C_{i|j}.$$

It is, however, remarked that (2.10)″ is solely a consequence of Proposition 4.

In (BC-III-3), we take another contraction for h and j to obtain

(2.11) $$P_{mk}{}^{(2)}|_i + P_{mr}{}^{(2)} C_k{}^r{}_i - S_{mi|k} + S_{mr} P^r{}_{ki}$$
$$= P_m{}^r{}_{ki}|_r + P_m{}^s{}_{ri} C_k{}^r{}_s - S_m{}^s{}_{ir} P^r{}_{ks}.$$

Lastly we are concerned with the Bianchi identities of the fourth group (BC-IV-1, 2). From (BC-III-1), as $C_{hji}|_k = C_{hik}|_j$ in the Cartan connection $C\Gamma$, it follows that

$$S_{hijk} = A_{(jk)}\{C_h{}^r{}_k C_{rij}\},$$

which is nothing but the well-known representation of the v-curvature tensor S_{hijk} of the Cartan connection $C\Gamma$ ([12](17.20), See §6). Substituting the above into (BC-IV-1, 2), these are automatically satisfied ([12] §7).

Here we shall return to (2.2)″. Because of $P_{or}{}^{(1)} = C_{r|o}$, if we put

(2.12) $$Z_i = P^s{}_{ir} R^r{}_{so} - R^r{}_{io} C_{r|o},$$

then (2.2)″ is written in the form

$$R_{oo|i} - R_{oi|o} - R^r{}_{oi|r} = Z_i,$$

so that we conclude

THEOREM 1. *In a Finsler space with* $Z_i=0$ *the tensor* $B_i^h=R^h{}_{io}+R_{oo}\delta_i^h-R_{oi}y^h$ *satisfies the conservation law*[2] $B^h{}_{i|h}=0$.

REMARK 5. Although the assumption $Z_i=0$ seems not to be natural, this is identically satisfied in a Finsler space of scalar curvature, as will be proved in §5 Th. 12. In a Riemannian space the above conservation law $B^h{}_{i|h}=0$ is a consequence of $R_i{}^r{}_{jk|r}=R_{ij|k}-R_{ik|j}$.

Next we shall be concerned with the conservation law with respect to the v-covariant differentiation. (See §6 as to the v-curvature tensor S_{hijk}). The equation (2.7) is notable in this point of view. Suppose that $2R_{oi}+R_{io}=0$. Then we get $R_{oo}=0$ and (2.7) reduces to $R^h{}_{io}|_h=0$. Consequently

THEOREM 2. *In a Finsler space with* $B_i=2R_{oi}+R_{io}=0$ *the contracted* (v) h*-torsion tensor* $R^h{}_{io}$ *satisfies the conservation law* $R^h{}_{io}|_h=0$.

REMARK 6. In a Finsler space of scalar curvature we have $R_{oi}=R_{io}$ (See §5 Th. 9(3)). Hence the tensor B_i in Theorem 2 is written in the form

$$B_i=3R_{io}=(n-2)L^2K_{\|i}+3(n-1)KLl_i.$$

Contracting the above by y^i we get $B_o=3(n-1)KL^2$. Consequently $B_i=0$ is equivalent to $K=0$ and Theorem 2 is trivial.

Consider the above condition $B_i=0$. It follows from (2.1) that the contracted Ricci tensors R_{oi} and R_{io} are expressible in

$$R_{oi}=-R_{io}/2=-(R_{\gamma io}+C_i{}^s{}_r R^r{}_{os})/3, \quad (R_{\gamma io}=C^m R_{mio}). \tag{2.13}$$

From the above result and (2.6)″ we have

COROLLARY. *In a Finsler space, where the contracted Ricci tensors* R_{oi} *and* R_{io} *are written in the form* (2.13), *the contracted* $(v)h$*-torsion tensor* $R^h{}_{io}$ *satisfies the conservation law* $R^h{}_{io}|_h=0$ *and* $C_{i|o|o}$ *is written in the form*

$$C_{i|o|o}=(R_{\gamma io}-2C_i{}^s{}_r R^r{}_{os})/3.$$

§3. *C*-reducible Finsler spaces and Ricci tensors.

In the present section we are concerned with C-reducible Finsler spaces. Because, for instance, the Randers space (See §6), which is important in the theoretical physics, is certainly C-reducible ([12] §36, [19], [27]).

DEFINITION. A non-Riemannian Finsler space F^n $(n\geqq 3)$ is called *C-reducible* ([7]), if the hv-torsion tensor C_{ijk} is written in the form

$$C_{ijk}=(C_ih_{jk}+C_jh_{ki}+C_kh_{ij})/(n+1). \tag{3.1}$$

It is well known that the v-curvature tensor S_{ijhk}, the $(v)hv$-torsion tensor

2) See [12] §26, [18].

P_{ijk} and the hv-curvature tensor P_{hijk} of F^n are respectively written in the following concrete form ([12] §30):

$$S_{ijhk}=A_{(ij)}\{h_{ik}C_{jh}+h_{jh}C_{ik}\}/(n+1)^2, \tag{3.2}$$

$$P_{ijk}=G_ih_{jk}+G_jh_{ki}+G_kh_{ij}, \tag{3.3}$$

$$P_{hijk}=N_{hi}h_{jk}+A_{(hi)}\{h_{ij}N_{kh}{}^{(1)}+h_{ik}N_{jh}{}^{(2)}\}, \tag{3.4}$$

where h_{ij} is the angular metric tensor and

$$C_{ij}=C^2h_{ij}/2+C_iC_j, \quad (C^2=C_iC^i), \quad G_i=C_{i|o}/(n+1),$$
$$N_{ij}{}^{(1)}=(C_{i|j}-C_iG_j-\mu h_{ij}/2)/(n+1), \quad (\mu=C^iG_i),$$
$$N_{ij}{}^{(2)}=(C_{i|j}+C_jG_i+\mu h_{ij}/2)/(n+1),$$
$$N_{ij}=-N_{ij}{}^{(1)}+N_{ji}{}^{(1)}=-N_{ij}{}^{(2)}+N_{ji}{}^{(2)}.$$

REMARK 7. From (3.3) it is observed that the C-reducible Finsler space is *P-reducible* ([13]). The converse does not hold good in general. Recently C. Shibata has showed ([21]) that in case of a Finsler space of scalar curvature the converse is correct.

From (3.4) the hv-Ricci tensors $P_{hk}{}^{(1)}$ and $P_{hk}{}^{(2)}$ are especially written as follows:

$$\begin{aligned} P_{hk}{}^{(1)}=&\{nC_{k|h}-C_{h|k}+L^{-1}(n+1)(G_kl_h+G_hl_k)+2C_hG_k \\ &-(n-1)C_kG_h-\varepsilon h_{hk}\}/(n+1), \quad (\varepsilon=C^r{}_{|r}+(n-1)\mu), \\ P_{hk}{}^{(2)}=&\{nC_{k|h}-C_{h|k}+L^{-1}(n+1)(G_kl_h+G_hl_k)-2C_kG_h \\ &+(n-1)C_hG_k-(\varepsilon-2(n-1)\mu)h_{hk}\}/(n+1). \end{aligned} \tag{3.5}$$

From (3.5) the *hv-scalar curvatures* $P^{(1)}$ $(=P_{hk}{}^{(1)}g^{hk})$ and $P^{(2)}$ $(=P_{hk}{}^{(2)}g^{hk})$ are written in the form

$$P^{(1)}=-P^{(2)}=-(n-2)\mu. \tag{3.6}$$

Suppose that $P^{(1)}=0$, i.e., $\mu=C^iG_i=0$. As $P_{ijk}=C_{ijk|o}$, it is remarked that $G_i=P_i/(n+1)$, where $P_i=P_i{}^m{}_m$.

Hence we have

THEOREM 3. *In a C-reducible Finsler space the hv-scalar curvature* $P^{(1)}$ *(or* $P^{(2)}$*) vanishes if and only if the vector* P_i $(=P_i{}^m{}_m)$ *is orthogonal to the torsion vector* C_i $(=C_i{}^m{}_m)$.

If we put $A_i=LC_i$ and $A^2=A_iA^i$ $(A^i=g^{ir}A_r)$, it is easily seen that

$$\mu=C^rC_{r|o}/(n+1)=(1/2)L^{-1}A^2{}_{|o}/(n+1).$$

Consequently we have

COROLLARY. *In a C-reducible Finsler space with constant $A^2(=A_iA^i)$, the hv-scalar curvatures $P^{(1)}$ and $P^{(2)}$ vanish.*

REMARK 8. This suggests us that the hv-curvature tensor P_{hijk} will play an important role in the investigation of Finsler spaces with constant A^2. Pay attension to the well-known equation $A_i=L\dot{\partial}_i(\log\sqrt{g})$ and Deicke's theorem [4].

From (3.4) and (3.5) we obtain

$$\begin{aligned} &P_{hk}{}^{(1)}-P_{kh}{}^{(1)}=P_{hk}{}^{(2)}-P_{kh}{}^{(2)}=(n+1)N_{hk}=A_{\langle hk\rangle}\{C_{k|h}+C_hG_k\},\\ (3.7)\qquad &N_{io}{}^{(1)}=N_{io}{}^{(2)}=G_i,\quad N_{oi}{}^{(1)}=N_{oi}{}^{(2)}=0,\\ &N_{io}=-N_{oi}=-G_i. \end{aligned}$$

THEOREM 4. *If the hv-Ricci tensor $P_{ij}{}^{(1)}$ (or $P_{ij}{}^{(2)}$) of a C-reducible Finsler space is symmetric, then the Finsler space is a Berwald space.*

Proof. If the tensor $P_{ij}{}^{(1)}$ (or $P_{ij}{}^{(2)}$) is symmetric, from (3.7) we have $N_{ij}=0$ and $N_{io}=0$. Consequently from (3.7) $G_i=C_{i|o}/(n+1)=0$. Hence from Lemma[3] it is concluded that the Finsler space is a Berwald space.

PROPOSITION 5. *If the symmetric part $P_{(hk)}{}^{(1)}$ (or $P_{(hk)}{}^{(2)}$) of the hv-Ricci tensor is written in the form*

$$P_{(hk)}{}^{(1)}\ (\text{or } P_{(hk)}{}^{(2)})=\lambda_1h_{hk}+\lambda_2g_{hk} \tag{3.8}$$

as a linear combination of the angular metric tensor h_{hk} and the fundamental tensor g_{hk} with the scalar coefficients λ_1,λ_2, then $C_{i|o}=0$ holds necessarily.

Proof. Suppose that $P_{(hk)}{}^{(1)}=\lambda_1h_{hk}+\lambda_2g_{hk}$. The contraction of this by y^k yields

$$P_{(ho)}{}^{(1)}=\lambda_2y_h.$$

Contracting the above by y^h, we have $\lambda_2=0$ and $P_{(ho)}{}^{(1)}=0$. Thus (1) of Proposition 3 gives $C_{i|o}=0$.

Consequently from Lemma quated above and Proposition 5, it is concluded that

THEOREM 5. *If, in a C-reducible Finsler space, the symmetric part $P_{(hk)}{}^{(1)}$ (or $P_{(hk)}{}^{(2)}$) of the hv-Ricci tensor is written as (3.8), then the Finsler space is a Berwald space.*

COROLLARY. *A C-reducible Finsler space is a Berwald space if one of the following conditions holds good:*

3) See [14] ; [12] Th. 30.4 : If a C-reducible Finsler space is a Landsberg space, then it is a Berwald space.

(1) $P_{(hk)}{}^{(1)}$ *(or* $P_{(hk)}{}^{(2)}$*) is proportional to* h_{hk}.

(2) $P_{(hk)}{}^{(1)}$ *(or* $P_{(hk)}{}^{(2)}$*) is proportional to* g_{hk}.

(3) $P_{(hk)}{}^{(1)}$ *(or* $P_{(hk)}{}^{(2)}$*)* vanishes.

REMARK 9. In a 2-dimensional Finsler space the hv-torsion tensor C_{ijk} is always written as (3.1) and

$$P_{ij}{}^{(1)}=P_{ij}{}^{(2)}\ (=P_{ij}),\quad P^{(1)}=P^{(2)}=0$$

hold because of P-symmetry ([12] § 28). See Remark 1.

THEOREM 6. *If the hv-Ricci tensor* P_{ij} *is symmetric in a 2-dimensional Finsler space* F^2*, then* F^2 *is a Landsberg space*[4].

Proof. The hv-curvature tensor P_{hijk} of F^2 is always written in the form

$$P_{hijk}=\sigma(l_h m_i - l_i m_h) m_j m_k,$$

where (l_i, m_i) is the *Berwald frame* ([12] § 28). The two unit vectors l_i and m_i are orthogonal to each other. Contracting the above equation by g^{ij} we have $P_{hk}{}^{(1)}=\sigma l_h m_k$. The hv-Ricci symmetry yields immediately $\sigma=0$ and we get $P_{hijk}=0$, which is equivalent to $P_{ijk}=0$. Then the proof is complete.

Next we shall consider the Bianchi identities of C-reducible Finsler space. From (3.1) and $S_{(ijk)}\{R_{ijk}\}=0$ the equation (1.9) is rewritten as

$$(1.9\text{C})\qquad R_{kh}-R_{hk}=[(n-2)R_{\gamma kh}-A_{(hk)}\{C_h R_{ok}+L^{-1}l_h R_{\gamma ok}\}]/(n+1).$$

The contraction of (1.9C) by y^h yields

$$(2.1\text{C})\qquad R_{ko}-R_{ok}=\{(n-1)R_{\gamma ko}+C_k R_{oo}\}/(n+1).$$

From (2.1C) we get the form of $R_{\gamma ok}(=-R_{\gamma ko})$ in terms of the h-Ricci tensor. Substituting this into (1.9C) we obtain

$$(1.9\text{C})'\qquad R_{kh}-R_{hk}+\frac{1}{(n-1)L}A_{(hk)}\{l_k(R_{ho}-R_{oh})\}$$
$$=\frac{1}{(n+1)}\{(n-2)R_{\gamma kh}+C_k T_h - C_h T_k\},$$

where we put

$$T_h=R_{oh}-L^{-1}R_{oo}l_h/(n-1).$$

Next we reconsider (2.2)′. Using (1) of Prop. 3 and (3.3), it is rewritten in the form

$$(2.2\text{C})'\quad R_{ok|i}-R_{oi|k}=R^r{}_{ki|r}+(n-2)G_r R^r{}_{ki}-A_{(ki)}\{L^{-1}G_r R^r{}_{ok}l_i+R_{ok}G_i\}.$$

---

4) See [12] § 25, [6].

The contraction of (2. 2C)′ by y^k yields

(2. 2C)″ $$R_{oo|i}-R_{oi|o}=R^r{}_{oi|r}+(n-1)R^r{}_{oi}G_r-R_{oo}G_i.$$

Next we are concerned with (2. 5)′. First from (3. 3) we derive

$$P^r{}_{ki|r}=G^r{}_{|r}h_{ki}+G_{i|k}+G_{k|i}-L^{-1}(G_{i|o}l^k+G_{k|o}l_i).$$

By virtue of the above, (3. 1) and (3. 3), the equation (2. 5)′ is rewritten, after long computation, in the form

(2. 5C)′ $$R_{ok}|_i+C_{i|o|k}=S_{(ik)}\{G_{i|k}-L^{-1}G_{i|o}l_k+(n-3)G_kG_i/2$$
$$-(R_{ok}-L^{-1}R_{oo}l_k)C_i/(n+1)\}+\nu h_{ik}+U_{ik},$$

where $S_{(ik)}$ means the interchange of indices i, k and summation and we put

(3. 9)
$$\nu=G^r{}_{|r}+(n-1)\zeta-R_{o\gamma}/(n+1),\quad \zeta=G^rG_r,$$
$$U_{ik}=R_{ik}-(R_{ik\gamma}+R_{\gamma ki}+L^{-1}l_iR_{\gamma ok}+C_iR_{ok})/(n+1).$$

It is observed that the right hand side of (2. 5C)′ is symmetric except the term U_{ik}.

The contraction of (2. 5C)′ by y^k yields

(2. 5C)″ $$R_{oo}|_i+C_{i|o|o}=R_{oi}+R_{io}-(2R_{\gamma oi}+R_{oo}C_i)/(n+1).$$

From (2. 5C)′ we have

PROPOSITION 6. *In a C-reducible Finsler space, if the tensor U_{ij} given by* (3. 9) *is symmetric, then the tensor*

$$R_{oi}|_j+C_{j|o|i}$$

is symmetric.

REMARK 10. In a Riemannian space the term U_{ik} is, of course, equal to R_{ik} which is symmetric. Then $R_{oi}|_j+C_{j|o|i}$ is nothing but R_{ji}.

REMARK 11. In an h-isotropic Finsler space F^n (See § 4), the above U_{ij} is written in the form

$$U_{ij}=R_{ij}-Ry_jC_i$$

and R_{ij} is symmetric. Hence, if U_{ij} is symmetric, we get $C_i=0$ provided $R\neq 0$. Consequently F^n is a Riemannian space from (3. 1). In this case U_{ij} is nothing but the Ricci tensor R_{ij} in the Riemannian space.

REMARK 12. In a Finsler space of scalar curvature K (See § 5), it is observed that $R_{ij}=R_{ji}$ (See § 5 Th. 9(3)). And U_{ij} is written in the form

$$U_{ij}=R_{ij}+\frac{L^2}{3(n+1)}(K_{\|\gamma}h_{ij}+K_{\|i}C_j+K_{\|j}C_i)-C_i(L^2K_{\|j}/3+Ky_j).$$

If U_{ij} is symmetric, we have $K=0$. Conversely, in a space of vanishing scalar curvature K, $U_{ij}(=R_{ij})$ is always symmetric.

Contracting U_{ik} in (3.9) by y^i and y^k, we get respectively

$$U_{ok}=R_{ok},\quad U_{ko}=R_{ko}-(2R_{\gamma ok}+C_kR_{oo})/(n+1). \tag{3.10}$$

Next we shall consider another condition $U_{ok}=U_{ko}=0$. From (3.10) we have

$$R_{oi}=0,\quad R_{io}=2R_{\gamma oi}/(n+1). \tag{3.11}$$

Here we shall consider the equation (2.6)″, which is, from (3.1), written as

$$C_{j|o|o}=R^r{}_{jo}|_r-R_{oj}+(2R_{\gamma jo}-R_{oo}C_j)/(n+1). \tag{2.6C$''$}$$

From (3.11) the equation (2.5C)″ reduces to $C_{i|o|o}=0$ and (2.6C)″ also reduces to

$$R^r{}_{jo}|_r=R_{jo}. \tag{3.12}$$

Furthermore we are concerned with more stronger condition $U_{ij}=0$. In this case (2.5C)′ is written as follows:

$$C_{i|o|k}=G_{i|k}+G_{k|i}+(n-3)G_iG_k+\kappa h_{ik},\quad (\kappa=G^r{}_{|r}+(n-1)\zeta).$$

Hence, by virtue of $C_{i|o}=(n+1)G_i$, the above, which also means $G_{i|k}=G_{k|i}$, is rewritten in the form

$$G_{k|i}=\{(n-3)G_kG_i+\kappa h_{ki}\}/(n-1).$$

Contracting this by g^{ki} we get $\zeta=0$. Conseqently the above reduces to a little simple form

$$G_{k|i}=\{(n-3)G_kG_i+\xi h_{ki}\}/(n-1),\quad (\xi=G^r{}_{|r}). \tag{3.13}$$

If the metric is positive-definite, $\zeta=0$ implies $G_i=0$.

Thus, making a summary of the results obtained above, we have

THEOREM 7. *In a C-reducible Finsler space with* $U_{ok}=U_{ko}=0$ *the following hold good*:

(1) *The contracted Ricci tensors* R_{oi} *and* R_{io} *satisfy the equation* (3.11),
(2) $C_{i|o|o}=0$,
(3) *the contracted* $(v)h$*-torsion* $R^r{}_{jo}$ *satisfies the equation* (3.12).

COROLLARY. *In a C-reducible Finsler space with* $U_{ij}=0$, *the conditions* (1), (2) and (3) *of Theorem* 7 *hold good and further*

(4) *the tensor* $G_{k|i}$ *is symmetric and written as* (3.13),
(5) *the space is a Berwald space, provided the metric be positive-definite.*

Here we shall recollect the Proposition 4. In virtue of $h_k{}^i|_i=-(n-1)L^{-1}l_k$ and $G_i|_o=0$, it follows from (3.3) that

$$P_i{}^r{}_j|_r=G^r|_r h_{ij}-S_{(ij)}\{nL^{-1}G_i l_j-G_i|_j\}.$$

On the other hand from (3.1) we have

$$C_i{}^r{}_{j|r}=C^r{}_{|r}h_{ij}/(n+1)+\frac{1}{(n+1)}S_{(ij)}\{C_{i|j}-(n+1)L^{-1}l_iG_j\}.$$

Consequently (2.9) is of the form

$$C_{i|o}|_j-C_{i|j}=\omega h_{ij}-S_{(ij)}\{(n-1)L^{-1}l_iG_j-G_i|_j+C_{i|j}/(n+1)\},$$
$$(\omega=G^r|_r-C^r{}_{|r}/(n+1)),$$

which is rewritten as

$$C_{i|o}|_j-C_{i|j}=(n+1)\ \{\omega_1 h_{ij}-L^{-1}(l_iG_j+l_jG_i)\},\quad (\omega_1=\omega/(n-1)). \tag{2.9C}$$

Hence we have

PROPOSITION 4C. *In a C-reducible Finsler space the symmetric tensor* $C_{i|o}|_j-C_{i|j}$ *is given by* (2.9C).

§ 4. Isotropic Finsler spaces.

We shall conider an *h-isotropic* Finsler space F^n which is introduced by H. Akbar-Zadeh ([1], [12] § 22).

In the isotropic Finsler space the h-curvature tensor R_{hijk} is written in the form

$$R_{hijk}=R(g_{hj}g_{ik}-g_{hk}g_{ij}). \tag{4.1}$$

THEOREM (H. Akbar-Zadeh) *An h-isotropic Finsler space of dimension* $n\geqq 3$ *are such that*

(1) $R=$*constant.*

(2) *P-symmetry and* $S_{hijk}=0$, *provided* $R\neq 0$.

From (4.1) it is easily verified that the Ricci tensor R_{ij} is symmetric, i.e., $R_{ij}=(n-1)Rg_{ij}$. The equations (2.2), (2.2)′, (2.2)″ are trivial because of $R=$constant and metrical property ($g_{ij|k}=0$) of the Cartan connection $C\Gamma$.

Here we shall reconsider (2.5)″ and (2.6)′ which were derived from the Bianchi identities of the second group. From (4.1) we have

$$R_{ijk}=R(y_jg_{ik}-y_kg_{ij}).$$

Consequently (2.5)″ reduces to

(2.5I)'' $$C_{i|o|o}=-RL^2C_i.$$

Next (2.6)' is now rewritten as

(2.6I)' $$C_{j|o|k}-C_{k|o|j}=R(y_jC_k-y_kC_j).$$

REMARK 13. The contraction of (2.6I)' by y^k yields (2.5I)'' immediately. From (2.5I)'' we see that $C_{i|o|o}=0$ is equivalent to $C_i=0$, provided $R\neq 0$. This fact has been shown already by H. Akbar-Zadeh ([1] p. 48).

REMARK 14. An h-isotropic and C-reducible Finsler space is a Riemannian space (See [14] Theorem 2).

Hereafter suppose that $R\neq 0$. Then we see $P_{ij}{}^{(1)}=P_{ij}{}^{(2)}$ because of the P-symmetry. We shall denote it by P_{ij}. The expressions (2.6) and (2.10) are written respectively as follows:

(2.6I) $$P_{lj|k}-P_{lk|j}=A_{(jk)}\{Rg_{lj}C_k+P_l{}^s{}_{jr}P^r{}_{ks}\},$$

(2.10I) $$P_{lj}|_k-P_{lk}|_j=P_l{}^s{}_{rk}\ C_s{}^r{}_j-P_l{}^s{}_{rj}\ C_s{}^r{}_k.$$

From these equations we get respectively

$$P^m{}_{i|m}=-(n-1)RC_i+P^m{}_rP_i{}^r{}_m,\quad (P^m{}_i=g^{mr}P_{ri}),$$

$$P^m{}_i|_m=-P^m{}_rC_i{}^r{}_m.$$

§ 5. The scalar curvature and Ricci tensors.

A Finsler space F^n is said to be of *scalar curvature* K ([2], [12] §26) if the equation

(5.1) $$R_{oioj}=KL^2h_{ij}$$

holds good at any (x,y) of F^n, and to be of *constant curvature* K if, furthermore, the scalar K is constant.

It is well-known ([12] (26.5)) that (5.1) is equivalent to

(5.2) $$R_{ijk}=L^2(K_{\|j}h_{ik}-K_{\|k}h_{ij})/3+K(y_jh_{ik}-y_kh_{ij}),$$

where we denot $_{\|i}=\partial/\partial y^i$.

In general the Berwald curvature tensor H_{hijk} satisfies the following identities ([12] §18):

(5.3) $$H_{hijk}=R_{ijk\|h}-2C_{irh}R^r{}_{jk},$$

(5.4) $$H_{hijk}=-H_{hikj},$$

(5.5) $$S_{(ijk)}\{H_i{}^h{}_{jk}\}=0.$$

Substituting (5.2) into (5.3), we have ([12] (26.6))

$$(5.6)\qquad H_{hijk}=K(g_{hj}g_{ik}-g_{hk}g_{ij})+(1/3)\ A_{(jk)}\{3K_{\|h}y_jg_{ik}+LK_{\|j}(2l_hg_{ik}-l_kg_{ih}-l_ig_{hk})+L^2K_{\|j\|h}h_{ik}\}.$$

Here we shall define three *H-Ricci tensors* from the Berwald curvature tensor H_{hijk} in the following form:

$$(5.7)\qquad H_{ij}=H_i{}^r{}_{jr},\quad H_{ij}{}^{(1)}=H^r{}_{irj},\quad H_{ij}{}^{(2)}=H^r{}_{rij}.$$

From (5.5) it is easily seen that $H_{ij}-H_{ji}=H_{ji}{}^{(2)}$.

Next it follows from (5.6) that

$$(5.8)\qquad H_{ij}=(n-1)(Kg_{ij}+K_{\|i}y_j+K_{\|j}y_i)+\{(n-2)L^2K_{\|i\|j}+(n+1)K_{\|i}y_j\}/3,$$

$$(5.9)\qquad H_{ij}{}^{(1)}=(n-1)Kg_{ij}+\{(n-3)(K_{\|j}y_i+K_{\|i}y_j)+L^2K_{\|m\|n}g^{mn}h_{ij}-L^2K_{\|i\|j}-(n+1)K_{\|i}y_j\}/3.$$

From (5.8), (5.9) the *H-scalar curvatures* H $(=H_{ij}g^{ij})$ and $H^{(1)}$ $(=H_{ij}{}^{(1)}g^{ij})$ are written in the form

$$(5.10)\qquad \begin{aligned} &H=n(n-1)K+(n-2)L^2K_{\|m\|n}g^{mn}/3,\\ &H^{(1)}=H. \end{aligned}$$

On the other hand, from (5.8), (5.9) it is observed that

$$(5.11)\qquad H_{ij}-H_{ji}=H_{ji}{}^{(1)}-H_{ij}{}^{(1)}=H_{ji}{}^{(2)}=(n+1)(K_{\|i}y_j-K_{\|j}y_i)/3.$$

THEOREM 8. *Let F^n $(n\geqq 3)$ be a Finsler space of scalar curvature K. Then F^n is of constant curvature if and only if the Ricci tensor H_{ij} (or $H_{ij}{}^{(1)}$) is symmetric or $H_{ij}{}^{(2)}$ vanishes.*

Proof. From (5.11) we have $K_{\|i}y_j-K_{\|j}y_i=0$. Contracting this by y^j, we get $K_{\|i}=0$, which means that K is a function of position only. By virtue of generalized Schur's theorem, K becomes constant (See [2], [12] Prop. 26.1). The converse is clear.

REMARK 15. This theorem as to H_{ij} has been shown by L. Berwald ([2] p. 775).

The relation between the Cartan's h-curvature tensor R_{hijk} and the Berwald's one H_{hijk} is generally given by

$$(5.12)\qquad R_{hijk}=(H_{hijk}-H_{ihjk})/2-P_{hrj}P_i{}^r{}_k+P_{hrk}P_i{}^r{}_j.$$

Next we shall treat the h-Ricci tensor R_{ij} in the space of scalar curvature. Substituting (5.6) into (5.12) we get

$$(5.13)\qquad R_{hijk}=K(h_{hj}h_{ik}-h_{hk}h_{ij})+A_{(jk)}\{h_{ik}M_{hj}-h_{hk}M_{ij}$$

$$-P_{hrj}P_i{}^r{}_k\},$$

where we put

$$M_{hj}(=M_{jh})=(L^2K_{\|h\|j}+3K_{\|h}y_j+3K_{\|j}y_h+6Kl_hl_j)/6.$$

Multiplying (5.13) with g^{hk} we have

$$(5.14)\qquad R_{ij}=\{(n-1)6Kg_{ij}+(3n-7)(K_{\|i}y_j+K_{\|j}y_i)+L^2K_{\|m\|n}g^{mn}h_{ij}$$
$$+(n-3)L^2K_{\|i\|j}\}/6+P_i{}^m{}_nP_j{}^n{}_m-P_i{}^m{}_jP_m.$$

From (5.14) the *h-scalar curvature* $R\ (=R_{ij}g^{ij})$ is given by

$$(5.15)\qquad R=\{3n(n-1)K+(n-2)L^2K_{\|m\|n}g^{mn}\}/3+g^{ij}P_i{}^m{}_nP_j{}^n{}_m-P^mP_m.$$

We have from (5.13) and (5.14) directly

THEOREM 9. *In a Fnisler space of scalar curvatvre K the following hold good*:

(1) $R_{hijk}=R_{jkhi}$,

(2) $S_{(ijk)}\{R_{hijk}\}=0$,

(3) *The h-Ricci tensor R_{ij} is symmetric and given by* (5.14).

REMARK 16. In a Finsler space of scalar curvature it is verified by means of (5.2) that

$$S_{(ijk)}\{C_h{}^r{}_iR_{rjk}\}=0,$$
$$A_{(jk)}\{C_i{}^r{}_jR_{rhk}-C_h{}^r{}_jR_{rik}\}=0.$$

Therefore from (1.7), (1.8) we have another proof of (1), (2) of Theorem 9 not refering to the components of the h-curvature tensor R_{hijk} of (5.13).

REMARK 17. In case of constant curvature, the Ricci tensor R_{ij} is, of course, symmetric as shown and used by Y. Takano [25]. In case of scalar curvature, (3) of Theorem 9 is shown independently and almost simultaneously by C. Shibata [21].

THEOREM 10. *Let $F^n(n\geqq 3)$ be a Finsler space of scalar curvature K. If $R_{ij}=\nu g_{ij}$ with some scalar ν, then F^n is of constant curvature.*

Proof. Suppose that $R_{ij}=\nu g_{ij}$. Contracting this by y^j, we get $R_{oi}=\nu y_i$. From (5.14) we have

$$R_{oi}=\{3(n-1)Ky_i+(n-2)L^2K_{\|i}\}/3.$$

These equations yield

$$3(n-1)Ky_i+(n-2)L^2K_{\|i}=3\nu y_i.$$

We take the contraction of the above by y^i to get $\nu=(n-1)K$, so that $K_{\|i}=0$ provided $n\geqq 3$. Consequently K is constant.

THEOREM 11. *Let F^n $(n\geqq 3)$ be a Finsler space of scalar curvature K. If $H_{(ij)}=\nu g_{ij}$ (or $H_{(ij)}^{(1)}=\nu' g_{ij}$) with some scalar ν (or ν'), then F^n is of constant curvature, where ( ) denotes the symmetric part of H_{ij} (or $H_{ij}^{(1)}$).*

This is easily obtained in the similar way to the proof of Theorem 10.

Here we recall Theorem 1 in §2. In case of scalar curvature, it is easily seen that the quantity Z_i (See (2.12)) is identically zero. Further. from (5.2), (5.14) the tensor B_i^h is written in the form

$$B_i^h=(n-2)L^2Z_i^h/3, \quad (Z_i^h=K_{\|i}y^h-3Kh_i^h, \ h_i^h=h_{ij}g^{hj}).$$

Hence we have from Theorem 1

THEOREM 12. *In an n $(\geqq 3)$-dimensional Finsler space of scalar curvature K, the tensor $Z_i^h=K_{\|i}y^h-3Kh_i^h$ satisfies the conservation law $Z_i^h{}_{|h}=0$.*

REMARK 18. In case of constant curvature K, from (BC-I-3) H. Rund has derived ([18] (3.15), [12] Th. 26.4) another conservational law $G_i^h{}_{|h}=0$, where

$$G^h{}_i=R^h{}_i-R\delta_i^h/2-KSy^hy_i/2, \quad (S=S_{ij}g^{ij}).$$

REMARK 19. In a Finsler space of scalar curvature K the equation, which is one of the Bianchi identities of the Berwald connection $B\Gamma$ ([12] (18.20)),

$$S_{(ijk)}\ \{G_m{}^h{}_{ir}R^r{}_{jk}+H_m{}^h{}_{ij;k}\}=0$$

reduces to a simple form

$$S_{(ijk)}\{H_m{}^h{}_{ij;k}\}=0.$$

This was suggested by Prof. M. Matsumoto. The semicolon (;) means the covariant differentiation with respect to the Berwald connection $B\Gamma$. The contraction of the above equation by g^{mk}, however, leads us rather complicated equation, because the Berwald connection $B\Gamma$ is not metrical.

Next we are concerned with a C-reducible Finsler space of scalar curvature K. In a C-reducible Finsler space, P_{ijk} is given by (3.3). Substituting this into (5.13), we obtain

$$R_{hijk}=A_{(jk)}\{h_{ik}N_{hj}+h_{hj}N_{ik}\}, \tag{5.16}$$

where we put $N_{hj}(=N_{jh})=(K-\zeta)h_{hj}/2+M_{hj}-G_hG_j$. The expression (5.16) is very interesting because the h-curvature tensor R_{hijk} is simply written in terms of an angular metric tensor h_{ij}. Cf. Theorem 29.2 of [12].

Similarly substituting (3.3) into (5.14), (5.15), we get respectively

(5.17) $R_{ij}=\{(n-1)(K-\zeta)+L^2K_{\|m\|}g^{mn}/6\}h_{ij}+(n-1)Kl_il_j$
$+\{(3n-7)L(l_iK_{\|j}+l_jK_{\|i})+(n-3)L^2K_{\|i\|j}\}/6-(n-3)G_iG_j,$

(5.18) $R=R_{ij}g^{ij}=n(n-1)K-(n-2)\{(n+1)\zeta-L^2K_{\|m\|n}g^{mn}/3\}.$

Consequently, in case of $K=0$, we have

THEOREM (H. Yasuda [25] Th. 7). *Let F^n ($n\geqq 3$) be a C-reducible Finsler space of vanishing scalar curvature K. Then F^n becomes locally Minkowski*[5] *if one of the following conditions holds good:*

(1) *In case of $n=3$, $R_{ij}=0$ and positive-definite.*

(2) *In case of $n\geqq 3$, $R=0$ and positive-definite.*

(3) *In case of $n>3$, $R_{ij}=0$.*

Proof. Suppose thak $K=0$. Then (5.17) and (5.18) reduces to respectively

(5.19) $$R_{ij}=-(n-1)\zeta h_{ij}-(n-3)G_iG_j,$$

(5.20) $$R=-(n-2)(n+1)\zeta.$$

If $R_{ij}=0$ and $n=3$, from (5.19) we obtain ζ $(=G^iG_i)=0$, so that $G_i=0$ (the Berwald space), provided the metric is positive-definite. Consequently we have $N_{hj}=0$, so that, from (5.16), $R_{hijk}=0$. If $R_{ij}=0$. and $n>3$, we contract (5.19) for i and j to obtain $\zeta=0$, so that we have $G_i=0$. The proof of (2) is similarly obtained from (5.20).

REMARK 20. The $(v)hv$-torsion tensor P_{ijk} takes place in (5.13), (5.14) and (5.15). We have treated these equations for C-reducible spaces where P_{ijk} is of a simple form (3.3). The simplest case $P_{ijk}=0$ leads us to a trivial result, because S. Numata ([17], [12] § 30) has shown that a Finsler space ($n\geqq 3$) of scalar curvature $K\neq 0$ with $P_{ijk}=0$ is a Rimannian space of constant curvature K.

§ 6. The v-curvature tensor S_{hijk} and the Ricci tensor S_{ij}.

We consider a tangent space $F_x{}^n$ of an n-dimensional Finsler space F^n at a point $x=(x^i)$. Then $F_x{}^n$ is regarded as an n-dimensional Riemannian space equipped with the fundamental tensor g_{ij} (x, y) where x is fixed. The components of the C-tensor $C_j{}^i{}_k$ are nothing but the Christoffel symbols constructed from g_{ij} (x, y) with respect to y and the v-curvature tensor S_{hijk} is the Riemannian curvature tensor of $F_x{}^n$. Consequently the tensor $U_i{}^j=S^j{}_i-\delta_i^jS/2$ $(S^j{}_i=g^{jr}S_{ri},\ S_{ij}=g^{mn}S_{imjn},\ S=g^{ij}S_{ij})$ satisfies the conservation law $U_i{}^j|_j=0$ which is the well-known result in Riemannian geometry [22].

The v-curvature tensor S_{hijk} and the v-Ricci tensor S_{ij} are studied by, for

5) If $R_{hijk}=0$ and $C_{ijk|h}=0$, then the Finsler space is called *locally Minkowski* ([12] § 24).

instance, F. Brickell [3], S. Kikuchi [5], M. Matsumoto [9], [11], [12], S. Numata [16] and others [15], [23], [24].

A Finsler space F^n is called (α, β)*-metric* if the fundamental function is of the form $L(\alpha, \beta)$, where $\alpha^2(x, y)=a_{ij}(x)y^iy^j$, $\beta(x, y)=b_i(x)y^i$ and $L(\alpha, \beta)$ is positively homogeneous of degree 1 in α, β. Here the quadratic form $\alpha^2(x, dx)$ is supposed to be a Riemannian metric of the space. If $L(\alpha, \beta)$ is of the form $L(\alpha, \beta)=\alpha+\beta$ (resp. α^2/β), the Finsler space is called the *Randers space* (resp. *Kropina space*). The concrete form of the v-curvature tensor S_{hijk} of the Randers space is seen in [8], [19]. The one of the Kropina space is given by C. Shibata [20]. Further in case of (α, β)-metric, which is called the *generalized Randers space*, S. Numata gives the v-curvature tensor S_{hijk} in a very simple form [16] and obtains the v-Ricci tensor S_{ij}. Here we sum up the results obtained hitherto related to the v-curvature tensor S_{hijk} and the v-Ricci tensor S_{ij}.

(I) In case of $n=2$

The v-curvature tensor S_{hijk} is identically zero ([10] p. 152, [12] Prop. 28 3).

(II) In case of $n=3$

(1) The v-curvature tensor S_{hijk} is always written in the following form [5], [11], [12]

(6.1) $$L^2S_{hijk} \parallel S(h_{hj}h_{ik}-h_{hk}h_{ij})$$

where S is some $(0)p$-homogeneous scalar field in y^i. Consequently the v-Ricci tensor S_{ij} is given by

(6.2) $$S_{ij}=L^{-2}Sh_{ij}.$$

(2) The v-curvature tensor S_{hijk} vanishes if and only if the v-Ricci tensor S_{ij} vanishes. Consequently the Finsler space F^3 with $S_{ij}=0$ is a Riemannian space under the well-known F. Brickell's conditions [3].

(3) If $R_{ij}=\nu g_{ij}$ with some scalar ν, then the v-curvature tensor S_{hijk} or the h-curvature tensor R_{hijk} vanishes [9].

(III) In case of $n\geqq 3$

(1) The v-curvature tensor S_{hijk} vanishes if and only if the indicatrix I_x is of constant curvature 1 ([12] Th. 31.1).

(IV) In case of $n=4$

(1) The v-curvature tensor S_{hijk} is always written in the form

(6.3) $$S_{hijk}=A_{(jk)}\{h_{hj}M_{jk}+h_{ik}M_{hj}\},$$

where $M_{ij}=S_{ij}-Sh_{ij}/4$, $S=S_{ij}g^{ij}$ ([11], [12] Th 31. 2).

(V) In case of $n\geqq 4$

(1) Let F^n be a Finsler space with (α, β)-metric. If $S_{ij}=0$, then F^n is a Riemannan space provided b^2 $(=a^{ij}b_ib_j)\neq$constant [16].

(2) Let the v-curvature tensor S_{hijk} is of the form (6. 1). Then the scalar $S=S(x)$ and the indicatrix I_x is of constant curvature $S+1$ ([12] Th. 31. 6).

(3) Suppose that the v-curvature tensor S_{hijk} is of the from

(6. 4) $$S_{hijk}=A_{(jk)}\{h_{hj}E_{ik}+h_{ik}E_{hj}\}$$

where E_{ij} is a $(-2)p$-homogenous Finsler tensor field. Then the followings hold good [15]:

(1°) E_{ij} is given by

$$E_{ij}=\frac{1}{n-3}\left\{\frac{S}{2(n-2)}h_{ij}-S_{ij},\right\}\quad (S=S_{ij}g^{ij}),$$

and (6. 4) is written in the form

(6. 4)′ $$S_{hijk}=\frac{S}{(n-2)(n-3)}(h_{hk}h_{ij}-h_{hi}h_{ik})-\frac{1}{(n-3)}A_{(ij)}\{h_{hk}S_{ij}-h_{hi}S_{ik}\}.$$

(2°) If $S_{ij}=0$, then $S_{hijk}-0$.

(3°) F^n is P-symmetry if and only if $S_{ij|0}=0$.

(VI) In case of $n\geqq 5$

(1) Let the v-curvature tensor S_{hijk} is of the form (6. 4). Then the following hold good:

(1°) The indicatrix I_x is conformally flat, i. e., the Weyl conformal curvature tensor vanishes [12].

(2°) The tensor

$$h_{ij}T_k+(n-2)(S_{ik}|_j+L^{-1}S_{ik}l_j)$$

is symmetric in j and k, where $T_k=S|_k/2+L^{-1}Sl_k$[15].

REMARK 21. In the paper [9], *quasi-C-reducible* Finsler spaces with $S_{ij}=0$ are also treated. Any Finsler spaces with (α, β)-metric is quasi-C-reducible. With respect to the contracted tensor $T_{ij}(=g^{mn}T_{imjn})$, S. Numata proved the following:

(1) Let F^n $(n\geqq 3)$ be a Finsler space with (α, β)-metric. If $T_{ij}=0$, then F^n is a Riemannian space provided $b^2\neq$constant, where the original tensor T_{hijk} is defined in the form

$$T_{hijk}=LC_{hij}|_k+l_hC_{ijk}+l_iC_{jkh}+l_jC_{khi}+l_kC_{hij}.$$

Recently this tensor T_{hijk} is used by Prof. M. Matsumoto in characterizing Finsler spaces with *cubic metric.*

References

[1] H. Akbar-Zadeh: *Les espaces de Finsler et certains de leurs généralisations.* Ann. Sci. École Norm. Sup **80** (1963), 1-79.

[2] L. Berwald: *Ueber Finslersche und Cartansche Geometrie. IV. Projektivkrümmung allgemeiner affiner Raüme und Finslersche Raüme skalarer Krümmung.* Ann. of Math. (2) **48** (1947), 755-781.

[3] F. Brickell: *A thorem on homogeneous functions.* J. London Math. Soc. **42** (1967), 325-329.

[4] A. Deicke: *Über die Finsler-Raüme mit $A_i=0$.* Arch. Math. **4** (1953), 45-51.

[5] S. Kikuchi: *Theory of Minkowski space and of non-linear connections in a Finsler space.* Tensor, N.S. **12** (1962), 47-60.

[6] G. Landsberg: *Über die Krümmung in der Variationsrechnung.* Math. Ann. **65** (1908), 313-349.

[7] M. Matsumoto: *On C-reducible Finsler spaces.* Tensor, N.S. **24** (1972), 29-37.

[8] M. Matsumoto: *On Finsler spaces with Randers' metric and special forms of important tensors.* J. Math. Kyoto Univ. **14** (1974), 477-498.

[9] M. Matsumoto: *On Einstein's gravitational field equation in a tangent Riemannian space of a Finsler space.* Rep. Math. Phy. **8** (1975), 103-108.

[10] M. Matsumoto: *Metrical differential geometry.* (Japanese) Kiso Sugaku Sensho **14**, Shokabo, Tokyo, 1975.

[11] M. Matsumoto: *On the indicatrices of a Finsler space.* To appear in Period Math. Hungar.

[12] M. Matsumoto: *Foundation of Finsler geometry and special Finsler spaces.* To appear in 1977 from Deutscher Verlag der Wissenschaften, Berlin.

[13] M. Matsumoto: *Finsler spaces with the hv-curvature tensor P_{hijk} of a special form.* To appear in Rep. on Math. Physno.

[14] M. Matsumoto and C. Shibata: *On the curvature tensor R_{ijkh} of C-reducible Finsler spaces.* J. Korean Math. Soc. **13** (1976), 21-24, 189.

[15] M. Matsumoto and H. Shimada: *On Finsler spaces with the curvature tensors P_{hijk} and S_{hijk} satisfying special conditions.* To appear in Rep. on Math. Phys.

[16] S. Numata: *On the curvature tensor S_{hijk} and the tensor T_{hijk} of generalized Randers spaces.* Tensor, N.S. **29** (1975), 35-39.

[17] S. Numata: *On Landsberg spaces of scalar curvature.* J. Korean Math. Soc. **12** (1975), 97-100.

[18] H. Rund: *Über Finslersche Raüme mit speziellen Krümmungseigenschaften.* Monatsh. Math. **66** (1962), 241-251.

[19] C. Shibata, H. Shimada, M. Azuma and H. Yasuda: *On Finsler spaces with Rander's metric.* Tensor, N.S. **31** (1977), 18-26.

[20] C. Shibata: *On Finster spaces with Kropina metric.* To appear, in Rep. on Math. Phys.

[21] C. Shibata: *On the curvature tensor R_{hijk} of a Finsler space of scalar curvature.*

To appear, in tensor N. S.
[22] J. L. Synge: *Relativity: The general theory.* North-Holland Pub. Comp. Amsterdam (1960).
[23] Y. Takano: *Gravitational field in Finsler spaces.* Lett. Nuovo Cimento (2) **10** (1974), 747-750.
[24] Y. Takano: *Variation principle in Finsler spaces.* Lett. Nuovo Cimento (2) **11** (1974), 486-490.
[25] Y. Takano: *On the theory of fields in Finsler spaces.* To appear in Internat. Sympos. Relativity and Unified Field Theory, Calcutta, 1975.
[26] H. Yasuda: *On Finsler spaces with absolute parallelism of line-element.* J. Korean Math. Soc. **13** (1976), 179-188.
[27] H. Yasuda and H. Shimada: *On Randers spaces of scalar curvature.* To appear in Rep. on Math. Phys.

Kushiro Technical College, Japan