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A CHARACTERIZATION OF THE UNITARY GROUPS $U_4(2^n)$ BY ONE CENTRAL INVOLUTION

BY SEUNG AHN PARK*

1. Introduction

The purpose of this paper is to generalize the main theorem in the unpublished paper [6].

Let $U_4(q)$ denote the four-dimensional projective unimodular unitary group, where $q=2^n$. Let H_q be the centralizer in $U_4(q)$ of a central involution of $U_4(q)$. Then the center of H_q is an elementary abelian subgroup of order q. The following is the main theorem in [6].

THEOREM (Suzuki): *Let G be a finite group. Suppose that G contains a subgroup H which satisfies the following two conditions*:

(1) *H is isomorphic to H_q, and*

(2) *$H=C_G(z)$ for any involution z in the center of H.*

Then one of the following holds:

(i) *H is normal in G, and $q-1$ is divided by $|G:H|$,*

(ii) *$q=2$ and $G=O(G)\cdot H$, or*

(iii) *G is isomorphic to $U_4(q)$.*

The geralization of the above theorem is done by omitting the assumption (2). Note that the center of H_q has $p-1$ involutions. Our theorem is as follows.

THEOREM: *Let G be a finite group, Suppose that G contains a central involution j such that the centralizer H of j in G is isomorpic to H_q. Then one of the following holds*:

(i) *H is normal in G,*

(ii) *$G=O(G)\cdot C_G(z)$ for some involution z in the center of H, or*

(iii) *G is isomorphic to $U_4(q)$.*

The structure of the group $U_4(q)$ has been studied in the author's paper [4]. We consider the subgroup T of H which corresponds to the maximal normal 2-subgroup of a parabolic subgroup of $U_4(q)$. If $N_G(T)$ is 2-closed, then we can show that the case (i) or (ii) of Theorem holds. If $N_G(T)$ is not 2-closed, then we can show that $N_G(T)/T$ is a (TI)-group and that

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$H=C_G(z)$ for any involution z in the center of H. Thus we can obtain the case (iii) of Theorem by using the same argument as that in [6].

We will use the same notation as that of [4] and omit the detailed discussion on the group $U_4(q)$. An element of order 2 in a group is called an *involution*. An involution is *central* if it is contained in the center of some Sylow 2-group. A finite group is *2-closed* if it has a normal Sylow 2-group. A finite group is called a *(TI)-group* if the intersection of any two distinct Sylow 2-groups is trivial. By $O(G)$ we mean the maximal normal subgroup of odd order in a finite group G.

2. Some Properties of H_q

Let F be a finite field with q^2 elements, where $q=2^n$. Denote α^q by $\bar{\alpha}$ for all α in F. Use the same notation as that of [4] to define

$$U_1=\{x_1(\alpha)\ :\ \alpha\in F\},$$
$$U_2=\{x_2(\beta)\ :\ \beta\in F, \bar{\beta}=\beta\},$$
$$U_3=\{x_3(\gamma)\ :\ \gamma\in F\},$$
$$U_4=\{x_4(\delta)\ :\ \delta\in F, \bar{\delta}=\delta\},$$
$$P=\{h_1(\mu)h_2(\lambda)\ :\ \bar{\mu}\mu=1,\ \bar{\lambda}\lambda^{-1}=1\},$$
$$S=U_1U_2U_3U_4.$$

Then the element $x_4(1)$ is a central involution of the group $U_4(q)$ and the centralizer H_q of $x_4(1)$ in $U_4(q)$ is

$$H_q=SP\cup SPn_2U_2$$

where n_2 is an involution.

Let G be a finite group which satisfies the condition of Theorem. Identify the involution j of G with $x_4(1)$, and the subgroup H of G with the centralizer H_q. Then H is of order $q^6(q^2-1)(q+1)$ and S is a Sylow 2-group of G. The multiplication in S is given by the commutator relations:

$$[x_1(\alpha), x_2(\beta)]=x_3(\alpha\beta)x_4(\alpha\bar{\alpha}\beta),$$
$$[x_1(\alpha), x_3(\gamma)]=x_4(\alpha\bar{\gamma}+\bar{\alpha}\gamma)$$

and all other types of commutators between elements of the various U_i are trivial. The subgroup P is an abelian group of order q^2-1 whose multiplication is defined by

$$h_i(\gamma)h_i(\delta)=h_i(\gamma\delta).$$

The subgroup SP is the normalizer of S in H, and the action of the element $h=h_1(\mu)h_2(\lambda)$ of P on S is given by

$$h^{-1}x_1(\alpha)x_2(\beta)h=x_1(\mu^{-2}\lambda\alpha)x_2(\lambda^{-2}\beta),$$

$$h^{-1}x_3(\gamma)x_4(\delta)h=x_3(\mu^{-2}\lambda^{-1}\gamma)x_4(\delta).$$

The involution n_2 transforms the elements of SP as follows:

$$n_2h_1(\mu)h_2(\lambda)n_2=h_1(\mu)h_2(\lambda^{-1}),$$

$$n_2x_1(\alpha)x_4(\beta)n_2=x_3(\alpha)x_4(\beta),$$

$$n_2x_2(\beta)n_2=x_2(\beta^{-1})h_2(\beta^{-1})n_2x_2(\beta^{-1}).$$

In particular

$$(n_2x_2(1))^3=1,$$

Denote $Z(H)$ by Z. It is easy to see that $Z=Z(H)=Z(S)=U_4$, and it is an elementary abelian group of order q. From the condition of Theorem it follows that $H=C_G(Z)=C_G(j)$ and that $H\subseteq C_G(z)$ for any involution z of Z. The maximal normal 2-subgroup of H is $D=U_1U_3U_4$. The extension of H over D splits. The subgroup of S which plays an important role in the following discussion is $T=U_2U_3U_4$. Let $Q=\{x_1(\alpha):\bar{\alpha}=\alpha\}\cdot T$ and $U=C_Q(x_1(1))$. The subgroup Q of S is of order q^5, and T and U are elementary abelian groups of order q^4 and q^3, respectively.

(2.1) *Any involution of S is contained in either T or D. If t is an element in $T-D$, then $C_H(t)$ is 2-closed with Sylow 2-group T. Any involution in $D-Z$ is conjugate in H to $x_3(1)$. Moreover, if t is in $(T\cap D)-Z$, then $C_H(t)$ is conjugate to Q by an element of P.*

Proof. This follows from an easy computation.

(2.2) *Any maximal elementary abelian subgroup of H is conjugate in H to either T or U.*

Proof. Let A be a maximal elementary abelian subgroup of H. By Sylow's theorem we may assume that $A\subseteq S$. By (2.1) either $A\subseteq D$ or A has an element t contained in $T-D$. Suppose that A has an element t contained in $T-D$. Then $A\subseteq C_S(t)=T$ by (2.1) and the maximality of A yields $A=T$. Suppose that $A\subseteq D$. Since D is normal in H, we may assume by (2.1) that $x_3(1)\in A$. Since $Q=C_S(x_3(1))$, we have $A\subseteq Q$. On the other hand, each involution of Q is contained in either T or U. By the maximality of A and the above argument, there is an involution u in $(U-T)\cap A$. This yields that $A\subseteq U=C_Q(u)$, and $A=U$ by maximality.

(2.3) *S is the only Sylow 2-group of H which contains T. Moreover, T is the only elementary abelian subgroup of S of order q^4.*

Proof. Since $N_H(S)=N_H(T)=SP$, the assertion follows from (2.2).

(2.4) $N_G(Z)=N_G(H)=\{x\in G : j^x\in Z\}$, *and the number* $|j^G \cap Z|=|N_G(Z) : H|$ *is odd.*

Proof. If $j^x\in Z$ then $H^x=C_G(j^x)\supseteq H$, which yields $H^x=H$. Since Z is the center of H and $N_G(Z)/H$ is of odd order, the assertion holds.

(2.5) *Two elements of* T *are conjugate in* G *only if they are conjugate in* $N_G(T)$.

Proof. Let u and v be two elements of T such that $v=u^x$ for some x in G. Since $C_G(v)$ contains both T and T^x, by Sylow's theorem there is an element y of $C_G(v)$ such that both T and T^{xy} are contained in the same Sylow 2-group of $C_G(v)$. But T is the unique maximal elementary abelian subgroup of order q^4 of a Sylow 2-group in G which contains T, by (2.3). Hence we have $T=T^{xy}$. Moreover, $v=u^x=u^{xy}$, and the assertion follows.

(2.6) *Any element* x *of* H *such that* $[x, T]\subseteq Z$ *is contained in* T. *In particular,* T *is a self-centralizing subgroup of* G.

Proof. The element x normalizes T. Hence $x\in N_H(T)=SP$. Now an easy computation yields $x\in T$.

(2.7) *If a Sylow 2-group* S_1 *of* G *contains* T, *then* S_1 *normalizes* T, *and* T *contains* $Z(S_1)$.

Proof. Since by (2.3) T is the unique elementary abelian subgroup of S_1 of order q^4, the group T is normalized by S_1. By (2.6) the center $Z(S_1)$ is contained in T.

(2.8) $N_G(S)=N_G(T)\cap N_G(Z)$

Proof. Since by (2.3) T is a characteristic subgroup of S, we have $N_G(S)\subseteq N_G(T) \cap N_G(Z)$. Let x be an element of $N_G(T)\cap N_G(Z)$. Then $T\subseteq S^x$ and $Z=Z$ (S^x). Hence $S^x\subseteq H=C_G(Z)$, and it follows from (2.3) that $S^x=S$. The assertion holds.

3. Nonsimple cases

For the remainder of this paper the symbols G, j, H S, P, Z, D, T and Q retain the meanings given to them in Section 2. This section is devoted to proving the following result:

(3.1) *If* $N_G(T)$ *is 2-closed, then one of the following holds*:
  (i) H *is normal in* G, *or*
  (ii) $G=O(G)\cdot C_G(z)$ *for some involution* z *in* Z.

We assume in this section that $N_G(T)$ is 2-closed and we proceed with the proof of (3.1). Note that the assumption yields $N_G(S)=N_G(T)\subseteq N_G(Z)$

by (2.8).

(3.2) *No element in* $H-Z$ *is conjugate in* G *to any element of* Z.

Proof. By (2.1) every involution of H is conjugate to an involution of T. By the assumption Z is normal in $N_G(T)$. Hence the result follows from (2.5).

(3.3) *If there is a conjugate* Z_1 *of* Z *such that* $|Z:Z_1\cap Z|=2$, *then the case* (ii) *of* (3.1) *holds.*

Proof. For some element x in G we have $Z_1=Z^x$. Let $I=Z_1\cap Z$ and $C=C_G(I)$. Then C contains both H and H^x. Since $Z^x\neq Z$, it follows from (2.4) that $C\neq H$ and $j\in Z-I$. It is also easy to see that $Z(C)=I$. Since $C_G(j)=H$ and $|Z:I|=2$, this yields that $C_G(z)=H$ for every element z in $Z-I$. Hence no element in $Z-I$ is conjugate in G to any element of I. In particular we have $j^G\cap Z\subseteq(Z-I)$. Consider the product z of all elements in $j^G\cap Z$. Since $|Z:I|=2$ and $|j^G\cap Z|$ is odd by (2.4), the element z is contained in $Z-I$, and it is an involution, By a Burnside lemma any two elements of Z are conjugate in G only if they are conjugate in $N_G(Z)$. Hence it is easy to see that $z^G\cap Z=\{z\}$. By (3.2) this implies that $\langle z\rangle$ is weakly closed in S with respect to G. Thus the result follows from a theorem of Glauberman [2].

(3.4) *If there is no conjugate* Z_1 *of* Z *such that* $|Z:Z_1\cap Z|=2$, *then the case* (i) *of* (3.1) *holds.*

Proof. By (2.4) it suffices to show that Z is normal in G. First, we will show that $C_G(t)$ is 2-closed for any element t in $T-D$. Let t be an element in $T-D$. Since $N_G(S)=N_G(T)$, it follows from (2.7) that S is the only Sylow 2-group of G which contains T. Hence $T=C_S(t)$ is a Sylow 2-group of $C_G(t)$. Suppose that $C_G(t)$ is not 2-closed. Then there is an element x in $C_G(t)$ such that $T^x\neq T$. Choose an element x in such a way that the intersection $I_1=T^x\cap T$ has a maximal order. Since $C_H(t)$ is 2-closed with Sylow 2-group T by (2.1), we have $H^x\neq H$. By (2.4) this implies that $Z^x\neq Z$. Set $I=Z^x\cap Z$. Then it follows from (3.2) that $I=I_1\cap Z=T^x\cap Z$, and the assumption of this proposition yields that $r=|T/I_1|=|Z/I|\geq 4$. Set $C_1=C_G(I_1)$. Then $C_1\subseteq C_G(t)$, and by the maximality of I_1 the group C_1/I_1 is a (TI)-group which is not 2-closed. Hence, by (4.2) of [5], there is a cyclic subgroup R of order $r-1$ in $N_G(T)\cap C_1$ which acts transitively on non-identity elements of T/I_1. Since $N_G(T)\subseteq N_G(Z)$, the subgroup R normalizes Z. Hence any element in $Z-I_1=Z-I$ has at least $r-1$ conjugates lying in Z, which implies $|Z:I|=r$ and $T=ZI_1$. Now we have $[T,R]\subseteq Z$. Hence $[T,R,S]=1$ and $[S,T,R]\subseteq Z$. By the three-subgroup lemma we have $[R, S, T] \subseteq Z$. This implies that $[R, S] \subseteq T$ by (2.6). Hence, by (5.3.6) of [3], it follows that $[S, R]=[S, R, R] \subseteq [T, R] \subseteq Z$. Again by the three-subgroup lemma, it follows, that R centralizes $[S, S]$. Since Z is contained in $[S, S]$, this is a contradiction. This proves that $C_G(t)$ is 2-closed.

Suppose that Z is not normal in G. Let x be an element of G such that $Z^x \neq Z$. Set $I=Z^x \cap Z$ and $C=C_G(I)$. Consider an element t in $T-D$. Then the involution j is contained in $Z-I$ by (2.4) and it is not conjugate in C to t^x by (3.2). By (4.1) of [5] there is an involution w in C which commutes with j and t^x, and wj is conjugate in C to either j or t^x, By the earlier argument $C_G(t^x)$ is 2-closed with Sylow 2-group T^x, which implies that w belongs to $T^x \cap H$. Suppose that wj is conjugate in C to j. Since wj must be in Z by (3.2), the involution w is in $T^x \cap Z = Z^x \cap Z$. Hence wj and j are contained in $Z-I$, and they are conjugate in $N_G(Z)$. But since $N_G(Z)/H$ is of odd order, this yields a contradiction. Now suppose that wj is conjugate in C to t^x. Then wt^x is conjugate in C to j, and wt^x is contained in Z^x by (3.2). Thus the involution w is contained in $T^x - D^x$, and $C_G(w)$ is 2-closed with Sylow 2-group T^x. This yields $j \in T^x \cap Z = Z^x \cap Z = I$. But this is a contradiction.
This completes the proof of (3.4).

(3.5) *The proposition* (3.1) *holds.*

Proof. This follows from (3.3) and (3.4).

4. Identification With $U_4(q)$

In order to complete the proof of our Theorem we will prove the following result:

(4.1) *If $N_G(T)$ is not 2-closed, then G is isomorphic to $U_4(q)$.*

We assume in this section that $N_G(T)$ is not 2-closed, and we try to obtain a condition for H so that we can apply the Suzuki's theorem to G.

(4.2) *$N_G(Q)$ is 2-closed with Sylow 2-group S.*

Proof. As in (3.13) of [4] we can show that Z is characteristic in Q. Since T is the only maximal elementary abelian subgroup of Q of order q^4 by (2.3), it is characteristic in Q. These yields that $N_G(Q)$ normalizes both T and Z, and we have $N_G(Q) \subseteq N_G(S)$ by (2.8). Hence the assertion holds.

(4.3) *$N_G(T)/T$ is a (TI)-group.*

Proof. Let S_1 be a Sylow 2-group, $\neq S$, of $N_G(T)$. By (2.3) the group

$N_G(T)\cap C_G(Z(S_1))$ is 2-closed with Sylow 2-group S_1, and this yields that $C_S(Z(S_1))=S\cap S_1$. Since $Z(S_1)\subseteq T$, it follows from (2.1) that $S\cap S_1$ is either equal to T or conjugate to Q by an element of P. Here, we use the fact that $Q^h\cap Q$ is either Q or T for any element h of P. Since S/T is abelian, $S\cap S_1$ is normalized by both S and S_1. Hence it follows from (4.2) that $S\cap S_1$ is equal to T.

(4.4) *The extension of $N_G(T)$ over T splits.*

Proof. By the structure of S, the extension of S over T splits. Hence the result follows from a theorem of Gaschütz [1].

(4.5) *$N_G(T)$ contains subgroups L and K which satisfy the following properties:*

(i) *K is a complement of S in $N_G(S)$ containing P,*

(ii) *L is a complement of T in $N_G(T)$ containing K, and*

(iii) *L contains a normal subgroup L_0 isomorphic to $L_2(q^2)$.*

Proof. It is obvious that there is at least one subgroup K satisfying (i). By (4.4) there is a complement L of T in $N_G(T)$. Now it is easy to see that $S\cap L$ is a Sylow 2-group of L and that $N_G(S)\cap L=N_L(S\cap L)$. Hence a complement of $S\cap L$ in $N_L(S\cap L)$ is a complement of S in $N_G(S)$, and it is conjugate to K. Therefore, we may assume that L satisfies (ii). By (4.3) the group L is a (TI)-group with an elementary abelian Sylow 2-group of order q^2. Hence it follows from (4.2) of [5] that L contains a normal subgroup L_0 satisfying (iii).

(4.6) *The group K normalizes exactly two Sylow 2-groups S and S_1 of $N_G(T)$.*

Proof. This follows from the property of $L_2(q^2)$. See (4.3) of [5].

(4.7) *Any two involutions of Z are conjugate in G.*
In particular $H=C_G(z)$ for any involution z of Z.

Proof. Let S_1 be a Sylow 2-group of $N_G(T)$ defined in (4.6). Then $Z(S_1)$ is normalized by K, and it is contained in T by (2.7). By the structure of H, the subgroup $R=\{h_2(\lambda) : \bar{\lambda}\lambda^{-1}=1\}$ of P acts semi-regularly on $T-Z$. Since $Z(S_1)\neq Z$ and $|R|=q-1$, this implies that any two involutions of $Z(S_1)$ are conjugate. Hence the result follows.

(4.8) *The proposition* (4.1) *holds.*

Proof. By (4.7) we can repeat the argument in [6] to obtain the result. We remark that we can also use the Suzuki's theorem introduced in Section 1. Since $N_G(T)$ is not 2-closed, as in the proof of (4.7), there is a conjugate Z_1 of Z such that $Z_1 \neq Z$ and $Z_1 \subseteq T$. By (2.4) and a theorem of Glauberman [2], therefore, the cases (i) and (ii) of the Suzuki's theorem can not occur. Hence G must be isomorphic to $U_4(q)$.

Now the proof of our Theorem has been completed.

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Sogang University