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BETA ALTERNATIVES AND BETA SCORES TWO-SAMPLE RANK TEST

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1. Introduction

Let $X_1, \cdots, X_m$ and $Y_1, \cdots, Y_n$ be two random samples drawn from two populations having continuous cumulative distribution functions H and G and density functions h and g, respectively. Let $W=(W_1, \cdots, W_N)$, with $N=m+n$ and

$$W_1 < W_2 < \cdots < W_N,$$

be the combined ordered (smallest to largest) statistic of the X' s and Y's, and let $Z=(Z_1, \cdots, Z_N)$ be the indicator random vector for the random sample Y' s in W, namely

$$Z_i = \begin{cases} 0 \text{ if } W_i \text{ is an } X, \\ 1 \text{ if } W_i \text{ is a } Y. \end{cases}$$

The distributions of Z as well as those of certain functions of Z have been studied by various authors for various alternatives.

Lehmann (1953) derived the distributions of Z and some linear functions of Z under the alternative hypothesis

$$H_L : H=F \text{ and } G=F^k.$$

This alternative includes, as special cases, the exponential and the Weibull distribution-the most commonly used models in life testing.

Savage (1956) continued the study of the distributions of Z under the Lehmann alternative and showed that it is possible to construct optimal critical regions of various types. He also introduced the statistic which is a locally most powerful rank test (LMPRT) statistic under H_L.

Davies (1971) compared some small and large sample properties of the Lehmann test with those of the test proposed by Savage for the two-sample problems and showed that the Lehmann test and the Savage test are asymptotically equivalent and optimal. Young (1973) considered the distribution of some two-sample rank-order statistic under H_L for the case when the sample is censored on the occurrence of a given order statistic.

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Gibbons (1964a) introduced the alternatives of the form

$$H_g^* : H=1-(1-F)^k, \quad G=F^k$$

and

$$H_g^{**} : H=1-(1-F)^{1+\theta}, \quad G=F^{1+\theta}, \quad \theta\geq 0.$$

If k is a positive integer greater than or equal to two, the alternative H_g^* is equivalent to the alternative that the X' s are distributed as the smallest of k random variables and the Y' s are distributed as the largest of these k random variables from some unspecified continuous distribution F. She (1964 b) also introduced the Psi test which is a LMPRT under H_g^*.

In this paper we will study the problem of testing the null hypothesis

$$H_0 : H=G$$

against the alternative hypotheses

$$H_\beta : H=F, \quad G=B(F\ ;\ a, b)$$

or

$$H_\beta^* : H=B(F\ ;\ c, d), \quad G=B(F\ ;\ a, b)$$

where $B(x\ ;\ a, b)$ is the standard Incomplete Beta distribution function, i. e.,

$$\begin{aligned} B(F(x)\ ;\ a, b) &= \sum_{t=a}^{a+b-1}\binom{a+b-1}{t}[F(x)]^t[1-F(x)]^{a+b-1-t} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^{F(x)} t^{a-1}(1-t)^{b-1}dt, \end{aligned}$$

and F denotes any continuous distribution function. Note that H_β^* reduces to H_β when $c=d=1$. We will call H_β and H_β^* the Beta alternative and the generalized Beta alternative, respectively. Notice also that the Lehmann alternative and the Gibbons alternative are special cases of H_β and H_β^*, respectively.

Capon (1961) showed, by using a theorem of Hoeffding (1951), that the LMPRT of H_0 against

$$H_1 : H=F_\theta, \quad G=F_\phi, \quad \theta, \phi\in R,$$

where F_θ is a specified family of distribution function (one for each θ), is based on a linear rank-order statistic. He also showed that the LMPRT of H_0 against H_1 is asymptotically efficient.

In this paper the Beta scores test which is a LMPRT under a Beta alternative is proposed. The properties of the Beta scores statistic will be investigated, including its exact and asymptotic distributions. Various rank tests are compared from the point of view of Pitman efficiency for normal and nonnormal distributions.

2. The distribution of rank-order statistics under beta alternatives.

Under the generalized Beta alternative, the probability function of Z is easily seen to be given by

$$P(z\,;\,H_\beta^*)=m!n!\int\cdots\int_{W_1<\cdots<W_N}\prod_{i=1}^{N}[h(w_i)]^{1-z_i}[g(w_i)]^{z_i}dw_1\cdots dw_N$$

$$=m!n!\int\cdots\int_{W_1<\cdots<W_N}\prod_{i=1}^{N}\left\{\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}[F(w_i)]^{c-1}[1-F(w_i)]^{d-1}\right.$$

$$\left.\cdot f(w_i)\right\}^{1-z_i}\left\{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}[F(w_i)]^{a-1}[1-F(w_i)]^{b-1}f(w_i)\right\}^{z_i}$$

$$dw_1\cdots dw_N$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int\cdots\int_{W_1<\cdots<W_N}\prod_{i=1}^{N}\left\{[F(w_i)]^{c-1}\right.$$

$$\left.\cdot[1-F(w_i)]^{d-1}\right\}^{1-z_i}\left\{[F(w_i)]^{a-1}[1-F(w_i)]^{b-1}\right\}^{z_i}f(w_i)\,dw_1\cdots dw_N.$$

Making the change of variables

$$u_i=F(w_i),\quad i=1,\cdots,\ N,$$

we have

$$P(z\,;\,H_\beta^*)=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int\cdots\int_{0<u_1<\cdots<u_N<1}$$

$$\cdot\prod_{i=1}^{N}[u_i^{c-1}(1-u_i)^{d-1}]^{1-z_i}[u_i^{a-1}(1-u_i)^{b-1}]^{z_i}du_1\cdots du_N$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int\cdots\int_{0<u_1<\cdots<u_N<1}$$

$$\cdot\ (u_1\cdots u_N)^{c-1}[(1-u_1)\cdots(1-u_N)]^{d-1}(u_1z_1\cdots u_Nz_N)^{a-c}$$

$$\cdot[(1-u_1)z_1\cdots(1-u_N)z_N]^{b-d}\ du_1\cdots du_N$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int\cdots\int_{0<u_1<\cdots<u_N<1}$$

$$u_1^{(c-1)+z_1(a-c)}\cdots u_N^{(c-1)+z_N(a-c)}(1-u_1)^{(d-1)+z_1(b-d)}$$

$$\cdots(1-u_N)^{(d-1)+z_N(b-d)}\,du_1\cdots du_N$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int\cdots\int_{0<u_1<\cdots<u_N<1}$$

$$u_1^{a_1}\cdots u_N^{a_N}\sum_{k_1=0}^{b_1}\cdots\sum_{k_N=0}^{b_N}(-1)^{k_1+\cdots+k_N}$$

$$\cdot\binom{b_1}{k_1}\cdots\binom{b_N}{k_N}u_1^{k_1}\cdots u_N^{k_N}du_1\cdots du_N,$$

where $a_i=(c-1)+z_i(a-c)$ and $b_i=(d-1)+z_i(b-d)$ for $i=1,\cdots,N$.

If we make a transformation to the new variables $V_1,\cdots V_N$ defined by

$$U_i=\prod_{j=i}^{N}V_j,\quad i=1,\cdots,N,$$

then the V's are distributed over the region $0<V_j<1$, $j=1,\cdots,N$. Since the Jacobian of the transformation is given by

$$|J|=v_2v_3^2\cdots v_N^{N-1}=\prod_{j=1}^{N}V_j^{j-1},$$

and since

$$U_1^{a_1}\cdots U_N^{a_N}=\prod_{j=1}^{N}V_j^{a_1+\cdots+a_j},$$

we obtain

$$P(z;H_\beta^*)=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\int_0^1\cdots\int_0^1|J|$$

$$\cdot\prod_{j=1}^{N}v_j^{a_1+\cdots+a_j}\left[\sum_{k_1=0}^{b_1}\cdots\sum_{k_N=0}^{b_N}(-1)^{k_1+\cdots+k_N}\right.$$

$$\left.\cdot\binom{b_1}{k_1}\cdots\binom{b_N}{k_N}\prod_{j=1}^{N}v_j^{k_1+\cdots+k_j}\right]dv_1\cdots dv_N$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\sum_{k_1=0}^{b_1}\cdots\sum_{k_N=0}^{b_N}$$

$$(-1)^{k_1+\cdots+k_N}\binom{b_1}{k_1}\cdots\binom{b_N}{k_N}$$

$$\prod_{j=1}^{N}\left[\int_0^1 v_j^{(j-1)+(a_1+\cdots+a_j)+(k_1+\cdots+k_j)}dv_j\right]$$

$$=m!n!\left[\frac{\Gamma(c+d)}{\Gamma(c)\Gamma(d)}\right]^m\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\sum_{k_1=0}^{b_1}\cdots\sum_{k_N=0}^{b_N}$$

(2. 1)

$$(-1)^{k_1+\cdots+k_N}\binom{b_1}{k_1}\cdots\binom{b_N}{k_N}\prod_{j=1}^{N}\frac{1}{j+a_1+\cdots+a_j+k_1+\cdots+k_j}$$

where again $a_i=(c-1)+z_i(a-c)$ and $b_i=(d-1)+z_i(b-d)$ for $i=1,\cdots,N$.

Under the Beta alternative H_β, that is when $c=d=1$, (2. 1) reduces to

$$P(z;H_\beta)=m!n!\left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\right]^n\sum_{k_1=0}^{(b-1)z_1}\cdots\sum_{k_N=0}^{(b-1)z_N}(-1)^{k_1+\cdots+k_N}$$

(2. 2)

$$\cdot\binom{(b-1)z_1}{k_1}\cdots\binom{(b-1)z_N}{k_N}$$

$$\cdot\prod_{j=1}^{N}\frac{1}{j+(a-1)(z_1+\cdots+z_j)+(k_1+\cdots k_j)}$$

Under the Lehmann alternative H_L, (2. 2) reduces to

$$P(z;H_L)=m!n!\ a^n\prod_{j=1}^{N}\frac{1}{j+(a-1)(z_1+\cdots+z_j)},$$

which is the same result as Savage's (1965, Cor. 7. a. 1).

The generalized Beta alternative $H_\beta{}^*$ involves four parameters $a, b, c,$ and d, and it is too general for applications. Thus, we are more inte rested in the Beta alternative H_β. Note that the distribution of rank orders under H_β is symmetric in the sense that

$$P(z;H_\beta)=P(z';H_\beta'),$$

where H_β' is the alternative hypothesis

$$H_\beta':\ H=F,\quad G=B(F;b,a),$$

and $z'=(z_N, z_{N-1},\cdots,z_1)$.

The formula (2. 2) was programmed on the IBM 360 to obtain the small sample exact probabilities of the rank orders under H_β for combinations of sample sizes $m=n=2,3,4,5$ and $a=b=2,3,4$ given in Table 5. The small sample probabilities of Z for other combinations can be found in Song (1975).

3. The beta scores test - LMPRT against H_β

Under the assumptions of Section 1, we are again given m X's and n Y's which are independent random samples from unknown continuous distributions H and G, respectively. It is desired to test

$$H_0 : H=G$$

against

$$H_\beta : H=F, \quad G=B(F;a,b).$$

In order to simplify the problem of finding the locally most powerful rank test of H_0 against H_β, we shall assume that

$$a-1=k(b-1)=k\theta,$$

where k is a constant and θ is a parameter.

The LMPRT of H_0 against H_β is derived from the following theorem (see Gibbons 1964a) for rank tests of the hypothesis $H_0 : H=G$ against alternatives of the form $H_a : H=u, G=Q(u,\theta),\ \theta\geq 0$.

THEOREM 3.1. *Let the distribution function $Q(u,\theta)$ satisfy the following regularity conditions:*

(i) For almost all u, the density function q of Q and $\partial q(u,\theta)/\partial\theta$ exist and are continuous with respect to θ in the interval $(0,\delta)$.

(ii) There exist functions $M_0(u)$ and $M_1(u)$, both integrable over $(0,1)$ and independent of θ, such that

$$|q(u,\theta)|\leq M_0(u), \quad |\partial q(u,\theta)/\partial\theta|\leq M_1(\theta), \quad \text{for } 0<\theta<\delta.$$

(iii) $Q(u,0)=u$.

Then the LMPRT of H_0 against $H_a : H=u, G=Q(u,\theta),\ \theta\geq 0$, is to reject H_0 when

$$\sum_{i=1}^{N} a_i z_i > c,$$

where $a_i=E[\partial q(U_i,\theta)/\partial\theta|_{\theta=0}]$, $U_1<U_2<\cdots<U_N$ are order statistics for a random sample of size N from the uniform distribution on $(0,1)$, and c is a constant determined by the size of the test.

Now, under H_β, $q(u,\theta)$ is a Beta density, and the regularity conditions in Theorem 3.1 are satisfied and we have

$$a_i=E\left\{\frac{\partial}{\partial\theta}\left[\frac{\Gamma((k+1)\theta+2)}{\Gamma(k\theta+1)\Gamma(\theta+1)}[U_i^k(1-U_i)]^\theta\right]\Big|_{\theta=0}\right\}$$

$$=2+E[k\ ln(U_i)+ln(U_{N-i+1})].$$

Making the change of variables

$$U_i=e^{-Y_{N-i+1}},\quad i=1,\cdots,N,$$

we have

$$a_i=2-[kE(Y_{N-i+1})+E(Y_i)],$$

where Y_i is the ith smallest observation in a random sample of size $N=m+n$ drawn from an exponential distribution having density $f(y)=e^{-y}$, $y>0$. From Epstein and Sobel (1953), we know that

$$E(Y_i)=\sum_{j=1}^{i}\frac{1}{N-j+1}$$

Hence

$$a_i=2-\left(\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1}\right).$$

Substituting these a_i's into the test criterion of Theorem 3.1, we conclude that the LMPRT against H_β is to reject H_0, if $\theta>0$, when

$$B_N=\sum_{i=1}^{N}\left(\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1}\right)Z_i<c.$$

Here the statistic B_N will be called the Beta scores statistic.

Notice that, when $k=0$, B_N reduces to the Savage (or exponential scores) statistic

$$S_N=\sum_{i=1}^{N}\left(\sum_{j=1}^{i}\frac{1}{N-j+1}\right)Z_i$$

which is the LMPRT against H_L. James (1967) compared several two-sample rank tests for scale changes in Gamma distributions, and he showed that the Savage test offers greater Pitman efficiency than some standard tests (for example, the Wilcoxon test) when the shape parameter is small.

When $k=-1$, B_N is equivalent to the Psi test statistic

$$\Psi_N=\sum_{i=1}^{N}[\Psi(N-i+1)-\Psi(i)]Z_i/m,$$

where $\Psi(x)=d\log\Gamma(x)/dx$. The Psi test, which was introduced by Gibbons (1964a, 1964b), is the LMPRT against $H_g{}^*$.

4. The distribution of B_N

In this section the exact and large sample distributions of B_N will be investigated. Since the probability of any $Z=(Z_1, \cdots, Z_N)$ under H_0 is given by $1/\binom{m+n}{n}$, the exact null distribution of B_N can be easily found for small values of m and n. The small sample exact distribution of B_N under H_β are given in Table 5 for selected parameters.

Exact moments of B_N under H_0 may be determined from the following lemma (Savage, 1956, pp. 604):

LEMMA 4.1. *Let $U=\sum_{i=1}^{m+n} a_i Z_i$ and $V=\sum_{i=1}^{m+n} b_i Z_i$, where a_i and b_i are constants. Then, under H_0, the exact mean and covariance of U and V are given by*

$$E(U)=n/N \sum_{i=1}^{N} a_i,$$

$$Cov(U, V)=\frac{mn}{N^2(N-1)}\left(\sum_{i=1}^{N} a_i b_i-\frac{1}{N}\sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i\right).$$

THEOREM 4.2. *Under the null hypothesis $H_0: H=G$, the mean and variance of B_N are*

$$E(B_N)=(k+1)n,$$

$$Var(B_N)=\frac{mn}{N(N-1)}\left[N(k-1)^2-(k^2+1)\sum_{j=1}^{N}\frac{1}{j}+2k\sum_{j=1}^{N}\sum_{i=1}^{j}\frac{j-i+1}{j(N-i+1)}\right].$$

Proof. Here, we have

$$a_i=\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1},$$

$$E(B_N)=\frac{n}{N}\left(\sum_{i=1}^{N}\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{i=1}^{N}\sum_{j=1}^{i}\frac{1}{N-j+1}\right)=(k+1)n.$$

Next, using the fact that

$$\left(\sum_{i=1}^{N} a_i\right)^2=[(k+1)N]^2,$$

$$\sum_{i=1}^{N}(a_i^2)=\sum_{i=1}^{N}\left(\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1}\right)^2$$

$$=\sum_{i=1}^{N}\left[\left(\sum_{j=1}^{N}\frac{k}{j}\right)^2+2\left(\sum_{j=i}^{N}\frac{k}{j}\right)\left(\sum_{j=N-i+1}^{N}\frac{1}{j}\right)+\left(\sum_{j=N-i+1}^{N}\frac{1}{j}\right)^2\right]$$

$$= (k^2+1)\sum_{i=1}^{N}\left(\sum_{j=i}^{N}\frac{1}{j}\right)^2+2k\sum_{q=1}^{N}\left(\sum_{j=q}^{N}\frac{1}{j}\right)\left(\sum_{i=N-q+1}^{N}\frac{1}{i}\right)$$

$$= (k^2+1)\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\min\{i,j\}}{ij}+2k\sum_{j=1}^{N}\sum_{i=1}^{j}\frac{j-i+1}{j(N-i+1)}$$

$$= (k^2+1)\,(2N-\sum_{j=1}^{N}\frac{1}{j})+2k\sum_{j=1}^{N}\sum_{i=1}^{j}\frac{j-i+1}{j(N-i+1)},$$

the variance is given by

$$\mathrm{Var}(B_N)=\frac{mn}{N^2(N-1)}\Big[N(k^2+1)\,(2N-\sum_{j=1}^{N}\frac{1}{j})$$

$$+2Nk\sum_{j=1}^{N}\sum_{i=1}^{j}\frac{j-i+1}{j(N-i+1)}-[N(k+1)]^2\Big]$$

$$=\frac{mn}{N(N-1)}\Big[N(k-1)^2-(k^2+1)\sum_{j=1}^{N}\frac{1}{j}+2k\sum_{j=1}^{N}\sum_{i=1}^{j}\frac{j-i+1}{j(N-i+1)}\Big].$$

The asymptotic normality of B_N under the null and alternative hypotheses will be proved by applying some theorems of Chernoff and Savage (1958). For any i which is a function of N, we have

$$\lim_{N\to\infty}a_i=\lim_{N\to\infty}\left(\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1}\right)$$

$$=k\int_0^{1-u}\frac{dx}{1-x}+\int_0^{u}\frac{dx}{1-x}$$

$$=-\log u^k(1-u),$$

where $u=\lim\limits_{N\to\infty} i/N$ and $0\leqslant u\leqslant 1$.

In the notation of Chernoff and Savage, let $H_m(x)$ and $G_n(x)$ be the empirical distribution functions of X's and Y's, respectively, and let

$$R(x)=\lambda_N H_m(x)+(1-\lambda_N)G_n(x),\quad \lambda_N=m/N.$$

Then the combined cumulative distribution function is given by

$$R_N(x)=\lambda_N H(x)+(1-\lambda_N)G(x).$$

Assume that for all N the inequalities $0<\lambda_0\leqslant\lambda_N\leqslant(1-\lambda_0)<1$ hold for some fixed $\lambda_0\leqslant 1/2$. Theorem 1 of Chernoff and Savage (1958, pp. 974) (which is slightly modified to suit our notations) states that the statistic

$$T_N=\int_{-\infty}^{\infty} J_N(R_N(x))\,dG_n(x),$$

where J_N need be defined only at $1/N,\ 2/N,\cdots,\ N/N$, but may have its domain of definition extended to $(0, 1]$ by a suitable convention, is asymptotically normally distributed if

(1) $J(R)=\lim_{N\to\infty} J_N(R)$ exists for $0<R<1$, and is not constant,

(2) $\int_{I_N} [J_N(R_N)-J(R_N)]dG_n(x)=o_p(N^{-1/2})$, where I_N is the interval in which $0<R_N(x)<1$,

(3) $J_N(1)=o(N^{1/2})$,

(4) $|J^{(i)}(R)|=\left|\frac{d^iJ}{dR^i}\right|\leqslant K|R(1-R)|^{-i-1/2+\delta}$ for $i=0, 1, 2$, and for some $\delta>0$.

In applying Theorem 1 of Chernoff and Savage we have some difficulty in the verification of condition (2). The following theorem, which is a simple extension of Theorem 2 of Chernoff and Savage, gives a suffcient condition under which conditions (1), (2), and (3) hold.

THEOREM 4.3. *If*

(1) $J_N(i/N)=\int_0^1 J(u)g_{i,N}(u)\,du$, *where* $g_{i,N}$ *is the density function of the ith order statistic from the uniform distribution on* $(0, 1)$,

(2) $|J^{(i)}(u)|\leqslant K[u(1-u)]^{-i-1/2+\delta},\ i=0, 1, 2,$

then the conditions (1), (2), *and* (3) *of Theorem* 1 *of Chernoff and Savage hold.*

The proof follows immediately from Theorem 2 of Chernoff and Savage.

If we define

$$J_N(i/N)=a_i=\sum_{j=1}^{N-i+1}\frac{k}{N-j+1}+\sum_{j=1}^{i}\frac{1}{N-j+1},$$

then we see that

$$B_N/n=\int_{-\infty}^{\infty} J_N[R_N(x)]dG_n(x),$$

$$J(R)=-\log R^k(1-R),\ \ 0<R<1.$$

To check the first condition of Theorem 4.3, we first notice that

$$\int_0^1 J(u) g_{i,N}(u)\, du$$

$$= \frac{N!}{(i-1)!(N-i)!}\left[k\int_0^1[-\log(1-u)]\ (1-u)^{i-1}u^{N-i}du + \int_0^1[-\log(1-u)]u^{i-1}(1-u)^{N-i}du\right]$$

$$= \frac{N!}{(i-1)!(N-i)!}\left[k\int_0^\infty y(e^{-y})^{i-1}\ (1-e^{-y})^{N-i}e^{-y}dy + \int_0^\infty y(1-e^{-y})^{i-1}(e^{-y})^{N-i}e^{-y}dy\right]. \tag{4.1}$$

Epstein and Sobel (1953) showed that the expected value of the ith smallest observation of a sample of size N from the exponential distribution is $\sum_{i}{}^{j}{}_{=1} 1/(N-j+1)$. Hence (4.1) reduces to

$$\sum_{j=1}^{N-i+1} \frac{k}{N-j+1} + \sum_{i=1}^{i} \frac{1}{N-j+1} = J_N(i/N).$$

The second condition of Theorem 4.3 can also be verified easily. Thus, using the notations

$$J(R) = -\log R^k(1-R),$$

$$R(x) = \lambda_N H(x) + (1-\lambda_N) G(x),$$

we may state the following theorem which can be easily obtained from Theorem 1 of Chernoff and Savage.

THEOREM 4.4. *The Beta scores statistic B_N has asymptotic normal distribution with the mean and variance*

$$\mu_N = n\int_{-\infty}^{\infty} J[R(x)]dG(x),$$

$$\sigma_N{}^2 = \frac{mn}{N}\left[\frac{n}{N} Var B(X) + \frac{m}{N} Var B^*(Y)\right],$$

where

$$Var\ B(X) = 2\iint_{-\infty<x<y<\infty} H(x)\ [1-H(y)]J'[R(x)]J'[R(y)]\ dG(x)dG(y),$$

$$Var\ B^*(Y) = 2\iint_{-\infty<x<y<\infty} G(x)[1-G(y)]J'[R(x)]J'[R(y)]\ dH(x)dH(y).$$

COROLLARY 4. 5. *Under the null hypothesis $H_0: H=G$, the mean and variance of the asymptotic distribution of B_N are*

$$\mu_N=(k+1)n,$$

$$\sigma_N^2=\frac{mn}{N}\left[(k+1)^2-\frac{k\pi^2}{3}\right].$$

Proof. From Corollary 2 of Chernoff and Savage, and from Theorem 4. 4, we have

$$\lim_{N\to\infty}\frac{N}{mn}\sigma_N^2=2\iint_{0<x<y<1}x(1-y)J'(x)J'(y)\,dxdy$$

(4. 2)
$$=\int_0^1 J^2(x)\,dx-\left[\int_0^1 J(x)\,dx\right]^2.$$

Since $J(x)=-\log x^k(1-x)$, (4. 2) can be written as

$$[2k^2+2k(2-\pi^2/6)+2]-(k+1)^2=(k+1)^2-\frac{k\pi^2}{3}.$$

5. Efficiency comparisons for various distributions

For comparing the large sample power of two sequences of tests, the concept of asymptotic relative efficiency (ARE) was introduced by Pitman (1949). The definition and theory used here follow those of Gibbons (1971).

Let T_n and T_n^* denote two consistent level α tests based on a random sample of size n for testing $H_0: \theta=\theta_0$ against the alternative $H_1: \theta\in\Omega-\{\theta_0\}$, where Ω is the parameter space. Let $\{\theta_i|i=1,2,\cdots\}$ be an infinite sequence in $\Omega-\{\theta_0\}$ such that $\lim_{i\to\infty}\theta_i=\theta_0$. Let $P(\theta|n)$ and $P^*(\theta|n)$ denote the power function of the two tests T and T^* based on T_n and T_n^*. Let γ be a constant such that $\alpha<\gamma<1$. Lastly, let $\{n_i\}$ and $\{n_i^*\}$ be two monotonically increasing sequences of positive integers such that

$$P(\theta_i|n_i)=P^*(\theta_i|n_i^*)=\gamma,\quad i=1,2,\cdots$$

Then the ARE of test T relative to test T^* is defined by

$$\text{ARE}(T,T^*)=\lim_{i\to\infty}n_i^*/n_i,$$

if this limit exists and is constant for all sequences $\{n_i\}$ and $\{n_i^*\}$.

In other words, the ARE is the inverse ratio of the sample sizes necessary to obtain any power γ for the tests T and T^*, while simultaneously the sample sizes approach infinity and the sequences of alternatives approach θ_0, and both tests have the same significance level.

Assuming that T_n and T_n^* are two tests satisfying the four regularity conditions of Gibbons (1971, pp. 278), the ARE of T relative to T^* is given by

$$\text{ARE}(T, T^*) = \lim_{n\to\infty} e(T_n)/e(T_n^*),$$

where

$$e(T_n) = \frac{[dE(T_n)/d\theta]^2|_{\theta=\theta_0}}{\sigma^2(T_n)|_{\theta=\theta_0}}$$

is called the efficacy of T_n.

The general distribution model of the scale problem for two independent random samples is

$$G(x) = H(\theta x),$$

where we can assume without loss of generality that the two distri butions have a common unknown median. The null hypothesis of identicaldistribution then is $H_0: \theta=\theta_0=1$ against either one-or two-sided alternatives. In this case, using the method in Capon (1961), the four regularity conditions of Gibbons can be easily verified. Here, since we assumed that X and Y population have the same median, we have $k=1$. For these scale problems with common median, the efficacy of B_N is given by (see, Song 1975)

$$\text{(5. 1)} \qquad e(B_N) = \frac{mn}{N(4-\pi^2/3)}\left[\int_{-\infty}^{\infty}\left[\frac{xh^2(x)}{H(x)} - \frac{xh^2(x)}{1-H(x)}\right]dx\right]^2.$$

In order to compare the Beta scores test with the F-test which is the best parametric test for normal-theory model, for variance differences, let $H(x)$ be the standard normal cumulative distribution function. Then, integrating by part, (5. 1) becomes

$$\text{(5. 2)} \qquad \frac{mn}{N(4-\pi^2/3)}\left[2+2\int_{-\infty}^{\infty} x^2h(x)\ \log\ H(x)\ dx\right]^2.$$

The parametric F-test, for normal distribution, has the efficacy of (see, Gibbons 1971, pp. 288)

$$e(F) = 2mn/N.$$

The integration in (5. 2) is performed numerically to have

$$\text{ARE}(B_N, F) \approx 1.00$$

Because of the lack of robustness of the F-test with departures from normality, it is of little interest to compare B_N with the F-test for dist ributions other than normal. Of more interest is the comparison of the nonparametric tests for various distributions as has been done by Klotz (1962). To compare the Beta scores test (B_N) with the normal scores test (K_N), Mood test (M_N) and Siegel-Tukey test (S-T) for the exponential, uniform, double exponential and Cauchy distributions, the efficacy expressions presented in Klotz (1962, pp. 500) are used. The results are summarized in Table 1.

For uniform distribution, the integral parts of the efficacy expressions for both the normal scores statistic and the Beta scores statistic give infinite values.

Table 1 Efficiency Comparisons for Different Densities

density	M_N/B_N	$S\text{-}_T/B_N$	K_N/B_N
exponential	0.78	0.47	∞
uniform	0.00	0.00	.
normal	0.76	0.61	1.00
double exp.	0.88	0.76	0.98
Cauchy	1.61	1.72	0.96

Table 1 indicates that the Beta scores test should be used in preference to the Siegel-Tukey test and the Mood test when the extreme rankings give more dispersion information than do the central rankings. On the other hand, the Siegel-Tukey and the Mood test should be used in preference for those distribution with heavy tails and more dispersion information in the central rankings.

6. Critical regions and power functions against H_β

The properties of B_N discussed in the previous sections suggest that they are more useful for testing the difference of dispersions of two populations. When $k=1$, the Beta scores weights are symmetric, and we shall call this Beta scores statistic the symmetric Beta scores statistic.

The critical values of the corresponding symmetric Beta scores test, for

Table 2 Critical Values for B_N when $k=1$ and $m=n$

$m=n$	$\alpha=0.01$	0.025	0.05	0.075	0.10
3	0.99	2.48	4.95	7.43	9.90
4	4.44	11.35	23.03	34.75	46.67
5	20.00	50.79	103.78	157.33	211.30
6	88.47	254.31	547.99	786.14	1026.90
7	385.44	981.25	1999.04	3037.64	4090.05
8	1662.05	4233.62	8622.96	13103.45	17647.93
9	7105.88	18106.66	36879.04	56019.25	75458.38
10	30173.60	76901.91	156618.41	237889.37	320340.53

small samples, can be obtained from Table 5 by summing up the values of B_N in the rejection region. For lerger samples, since the weights are symmetric, the scheme introduced by Klotz (1962) can be used to reduce the number of enumerations. For example, when $m=n=10$, the total number of orderings is $\binom{20}{10}=184,756$, and, if we use his scheme, the number of distinct values of B_N is 8,953 which is less than five percent of the total number of orderings. The scheme was programmed for a computer with $m=n=6(1)10$ to find the critical values of B_N. Table 2 gives selected randomized critical values of the symmetric Beta scores test for $m=n=3(1)$ 10 and for $\alpha=0.01$, 0.025, 0.05, 0.075 and 0.10.

To calculate the power of a rank test the probabilities under the alternative of those orderings which lie in the rejection region of the test are summed. The power functions of the most powerful rank test (MP), the Klotz normal scores test (K_N), and the Beta scores test (B_N), against H_β, are tabulated in Table 3 for $a=b=2,3,4$, and $m=n=2(1)5$.

Table 3 Power Functions against H_β

$m=n$	Test	$\alpha=0.01$			$\alpha=0.05$			$\alpha=0.10$		
		$a=b=2$	3	4	2	3	4	2	3	4
	MP	.015	.017	.019	.073	.085	.093	.146	.170	.186
2	B_N	.015	.017	.019	.073	.085	.093	.146	.170	.186
	K_N	.015	.017	.019	.073	.085	.093	.146	.170	.186
	MP	.018	.024	.028	.092	.121	.142	.185	.242	.283
3	B_N	.018	.024	.028	.092	.121	.142	.185	.242	.283
	K_N	.018	.024	.028	.092	.121	.142	.185	.242	.283
	MP	.027	.041	.053	.116	.169	.211	.215	.295	.355
4	B_N	.027	.041	.053	.114	.163	.201	.214	.294	.352
	K_N	.027	.041	.053	.114	.163	.201	.214	.294	.352
	MP	.034	.059	.082	.139	.216	.278	.247	.355	.433
5	B_N	.034	.059	.082	.139	.214	.275	.247	.355	.433
	K_N	.034	.059	.082	.139	.214	.275	.247	.355	.433

Table 3 shows that for small samples the locally most powerful rank test B_N has almost the same power as that of the most powerful rank test. It should also be mentioned that the power functions of the normal scores test and the symmetric Beta scores test are almost the same for each selected value of α.

7. Discussion and applications

The Beta alternative introduced in Section 1 is of the form

$$H_\beta : H=F, \quad G=B(F\ ; a, b),$$

where B is the standard Incomplete Beta distribution function, a and b are positive real numbers, and F denotes any continuous distribution function. This Beta alternative includes the Lehmann alternative as a special case. Wilcoxon, Rhodes and Bradley (1963) states: "The Lehmann alternative involves change in the shape of Y relative to that of X but a principal change is in location." The Beta alternative, which is a generalization of the Lehmann altenative, involves not only location problems but also scale problems.

An easy interpretation of the Beta alternative may be given in terms of $p=P(X\leq Y)$. Since we have

$$p=P(X\leq Y)=\int_0^1\int_0^y dx \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} y^{a-1}(1-y)^{b-1}dy=\frac{a}{a+b},$$

the parameter p which is associated with the parameters a and b measures the intensity with which Y population tends to be larger than X population.

For integer values of a and b, Y has the cumulative distribution function of the ath order statistic of $(a+b-1)$ observations from a population with the cumulative distribution function F. Hence, if the information of order statistics from the null distribution is available, we can have some idea on the distribution of Y. Some of the means and variances of order statistics from the standard normal distribution are available (e. g., Godwin 1947, Rubin 1954, Harter 1961, $\cdots$).

Harter's table is used to give, in Table 4, the mean μ and variance σ^2 of Y for different values of a and b when X has standard normal distribution. The missing values can be obtained from the following identities:

$$\mu(a, b)=-\mu(b, a), \quad \sigma^2(a, b)=\sigma^2(b, a).$$

Table 4 provides a rough guide to the selection of a and b for a given translation or dispersion alternatives. It is seen in the table that for a fixed b (a) the variance of Y decreases as $a(b)$ increases and we know also that the distributionof Y will be skewed relative to that of X. For $a=b$, X population and Y population have the same mean but the variance of Y decreases as a or b increases. This fact implies that we use Table 4 to select a and b for a given scale alternative with a mean, which is not possible for the case of Lehmann alternative.

Assuming that H and G are continuous and are the same in all respects except that they differ in scale parameters, we can set

$$H(x)=F\left(\frac{x-\mu}{\sigma_1}\right) \text{ and } G(x)=F\left(\frac{x-\mu}{\sigma_2}\right),$$

Table 4 The Mean and Variance of Y when X has Standard Normal Distribution

$a\diagdown b$	1	2	3	4	5	6	7	8	9	10
1	.000									
	1.000									
2	.546	.000								
	.682	.449								
3	.846	.297	.000							
	.560	.360	.287							
4	1.029	495	.202	.000						
	.492	312	.246	.210						
5	1.163	.642	.353	.153	.000					
	.448	.280	.220	.187	.166					
6	1.267	.757	.473	.275	.123	.000				
	.416	.257	.201	.171	.151	.137				
7	1.352	.852	.572	.376	.225	.103	.000			
	.392	.238	.186	.158	.140	.127	.117			
8	1.424	.932	.656	.462	.312	.191	.088	.000		
	.373	.226	.175	.148	.131	.118	.109	.102		
9	1.485	1.001	.729	.537	.388	.267	.165	.077	.000	
	.357	.215	.166	.140	.123	.117	.103	.096	.090	
10	1.539	1.062	.793	.603	.456	.335	.224	.146	.069	.000
	.344	.205	.158	.133	.117	.106	.097	.091	.085	.081

where σ_1 and σ_2 are scale parameters. Without loss of generality we have taken μ to be the common unknown median for H and G. We wish to test the null hypothesis

$$H_0 : \sigma_1^2/\sigma_2^2=1$$

against the alternative

$$H_S : \sigma_1^2/\sigma_2^2>1.$$

The joint rankings of the m X-observations and n Y-observations give the value of the symmetric Beta scores statistic B_N. If the values of B_N fall in the rejection region we reject H_0 and accept H_S. For example, suppose that $m=n=5$, and $\alpha=0.10$, we reject H_0 if $B_N<211.30$.

If the X's come from the uniform distribution on $(0,1)$, and the Y's come from the Beta distribution with the same a and b, the symmetric Beta scores test is the LMPRT. In other words, if a and b are close to 1, the B_N-test is the most powerful rank test for testing H_0 against H_S.

If the X's and Y's come from the normal distributions, the usual test of significance for testing H_0 is the variance ratio F-test, which is the most commonly used statistical test for comparing variances. Even though the F-test is the most powerful parametric test, it is too sensitive to some departure from normality and we thus prefer to use rank tests when we are not sure about the normality of the underlying populations.

8. Acknowledgement

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Table 5 Probabilities of Rank Orders under H_β

i	$R.O.$	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
		$(m=n=2)$			
1	0110	3.3333	.2429	.2835	.3096
2	0011	4.0000	.1714	.1755	.1826
3	0101	4.0000	.1571	.1450	.1348
4	1001	4.6667	.1000	.0714	.0556
		$(m=n=3)$			
1	001110	4.9500	.0923	.1208	.1417
2	010110	5.2000	.0744	.0830	.0859
3	001101	5.7500	.0549	.0546	.0532
4	000111	6.0000	.0532	.0576	.0613
5	001011	6.0000	.0495	.0485	.0472
6	010101	6.0000	.0440	.0370	.0317
7	011001	6.0000	.0436	.0363	.0308
8	010011	6.2500	.0394	.0323	.0274
9	100101	6.8000	.0257	.0161	.0112
10	100011	7.0500	.0229	.0138	.0095
		$(m=n=4)$			
1	00111100	6.3429	.0381	.0589	.0764
2	01111000	6.6762	.0316	.0454	.0567
3	01011100	6.6762	.0292	.0378	.0427
4	01110100	6.8095	.0275	.0351	.0397
5	01101100	6.8095	.0270	.0337	.0374
6	01011010	7.0095	.0224	.0241	.0237
7	01110010	7.1429	.0209	.0220	.0214
8	01101010	7.1429	.0206	.0213	.0205
9	01100110	7.2762	.0189	.0187	.0175

i	$R. O.$	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
10	10111000	7.5333	.0181	.0194	.0199
11	10011100	7.5333	.0165	.0157	.0144
12	11110000	8.0000	.0158	.0179	.0198
13	10110100	7.6667	.0157	.0149	.0138
14	10101100	7.6667	.0154	.0142	.0129
15	11011000	7.8667	.0154	.0158	.0160
16	11101000	8.0000	.0149	.0155	.0159
17	11010100	8.0000	.0134	.0121	.0110
18	11001100	8.0000	.0130	.0114	.0101
19	11100100	8.1333	.0128	.0117	.0108
20	10011010	7.8667	.0126	.0100	.0079
21	10110010	8.0000	.0119	.0093	.0074
22	10001110	8.0000	.0117	.0090	.0070
23	10101010	8.0000	.0117	.0089	.0070
24	10010110	8.0000	.0116	.0088	.0068
25	10100110	8.1333	.0107	.0078	.0059
26	11010010	8.3333	.0101	.0075	.0058
27	11001010	8.3333	.0099	.0071	.0055
28	11100010	8.4667	.0097	.0072	.0057
29	11000110	8.4667	.0090	.0062	.0046
30	10011001	8.7238	.0071	.0041	.0026
31	10110001	8.8571	.0067	.0038	.0024
32	10101001	8.8571	.0066	.0037	.0023
33	10100101	8.9905	.0061	.0032	.0020
34	11010001	9.1905	.0057	.0030	.0019
35	11001001	9.1905	.0055	.0029	.0018
36	11100001	9.3238	.0054	.0029	.0018
37	11000101	9.3238	.0051	.0026	.0015
38	11000011	9.6571	.0043	.0020	.0012
		$(m=n=5)$			
1	0011111000	7.8968	.01396	.02474	.03490
2	0011110100	8.0873	.01179	.01831	.02323
3	0011101100	8.1706	.01105	.01650	.02037
4	0101111000	8.2718	.01045	.01527	.01867
5	0111110000	8.4623	.01016	.01592	.02107
6	0110111000	8.4623	.00931	.01295	.01540
7	0111101000	8.5456	.00920	.01312	.01600

i	$R.O.$	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
8	0111011000	8.5456	.00898	.01242	.01477
9	0101110100	8.4623	.00881	.01127	.01237
10	0100111100	8.4623	.00874	.01106	.01203
11	0101101100	8.5456	.00825	.01011	.01079
12	0101011100	8.5456	.00823	.01006	.01070
13	0110110100	8.6528	.00784	.00953	.01015
14	0111100100	8.7361	.00771	.00953	.01034
15	0111010100	8.7361	.00755	.00909	.00967
16	0110101100	8.7361	.00733	.00852	.00880
17	0110011100	8.7361	.00729	.00842	.00867
18	0111001100	8.8194	.00703	.00808	.00832
19	0101110010	8.8373	.00653	.00679	.00638
20	0101101010	8.9206	.00613	.00615	.00564
21	0110110010	9.0278	.00580	.00572	.00521
22	1001111000	9.1607	.00579	.00615	.00606
23	1011110000	9.3512	.00570	.00658	.00711
24	0111100010	9.1111	.00568	.00567	.00523
25	0111010010	9.1111	.00557	.00544	.00494
26	0110101010	9.1111	.00544	.00516	.00459
27	0110011010	9.1111	.00542	.00513	.00455
28	0111001010	9.1944	.00521	.00488	.00431
29	1010111000	9.3512	.00519	.00528	.00509
30	1011101000	9.4345	.00515	.00541	.00537
31	1011011000	9.4345	.00502	.00510	.00493
32	1001110100	9.3512	.00488	.00523	.00400
33	1000111100	9.3512	.00481	.00438	.00381
34	0110100110	9.3016	.00481	.00430	.00368
35	1101110000	9.7262	.00473	.00516	.00543
36	0111000110	9.3849	.00460	.00405	.00344
37	1001101100	9.4345	.00456	.00405	.00347
38	1111100000	10.0000	.00455	.00540	.00622
39	1001011100	9.4345	.00454	.00401	.00343
40	1110110000	9.9167	.00438	.00474	.00499
41	1010110100	9.5417	.00437	.00387	.00334
42	1111010000	10.0000	.00433	.00479	.00516
43	1011100100	9.6250	.00431	.00362	.00346
44	1100111000	9.7262	.00428	.00409	.00382

i	$R.O.$	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
45	1101101000	9.8095	.00427	.00423	.00408
46	1011010100	9.6250	.00421	.00372	.00321
47	1101011000	9.8095	.00415	.00397	.00373
48	1010101100	9.6250	.00407	.00345	.00289
49	1010011100	9.6250	.00404	.00340	.00282
50	1110101000	10.0000	.00395	.00386	.00373
51	1011001100	9.7083	.00392	.00330	.00275
52	1111001000	9.7083	.00389	.00388	.00382
53	1110011000	10.0000	.00383	.00361	.00338
54	1001110010	9.7262	.00361	.00272	.00206
55	1100110100	9.9167	.00360	.00289	.00250
56	1000111010	9.7262	.00359	.00268	.00201
57	1101100100	10.0000	.00357	.00305	.00262
58	1101010100	10.0000	.00348	.00289	.00242
59	1001101010	9.8095	.00339	.00246	.00181
60	1001011010	9.8095	.00338	.00245	.00180
61	1100101100	10.0000	.00335	.00265	.00215
62	1100011100	10.0000	.00332	.00260	.00209
63	1110100100	10.1905	.00330	.00278	.00238
64	1111000100	10.2738	.00324	.00278	.00242
65	1101001100	10.0833	.00323	.00255	.00207
66	1010110010	9.9167	.00323	.00232	.00171
67	1110010100	10.1905	.00321	.00262	.00218
68	1000110110	9.9167	.00319	.00226	.00164
69	1011100010	10.0000	.00317	.00232	.00174
70	1000011110	10.0000	.00311	.00222	.00162
71	1011010010	10.0000	.00310	.00222	.00164
72	1000101110	10.0000	.00306	.00214	.00154
73	1010101010	10.0000	.00302	.00209	.00150
74	1010011010	10.0000	.00301	.00207	.00148
75	1001010110	10.0000	.00300	.00205	.00146
76	1001100110	10.0000	.00299	.00205	.00146
77	1110001100	10.2738	.00297	.00230	.00185
78	1011001010	10.0833	.00290	.00199	.00142
79	1001001110	10.0833	.00287	.00193	.00136
80	1010100110	10.1905	.00267	.00173	.00120
81	1010010110	10.1905	.00266	.00173	.00119

i	R. O.	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
82	1100110010	10.2917	.00265	.00179	.00127
83	1101100010	10.3750	.00262	.00181	.00132
84	1101010010	10.3750	.00256	.00172	.00123
85	1011000110	10.2738	.00256	.00164	.00113
86	1010001110	10.2738	.00254	.00162	.00111
87	1100101010	10.3750	.00248	.00160	.00111
88	1100011010	10.3750	.00246	.00158	.00109
89	1110100010	10.5655	.00242	.00164	.00119
90	1101001010	10.4583	.00239	.00153	.00106
91	1111000010	10.6488	.00238	.00163	.00120
92	1110010010	10.5655	.00236	.00155	.00110
93	1110001010	10.6488	.00220	.00138	.00095
94	1100100110	10.5655	.00219	.00133	.00089
95	1100010110	10.5655	.00218	.00132	.00088
96	1101000110	10.6488	.00210	.00127	.00084
97	1100001110	10.6488	.00208	.00123	.00081
98	1001110001	10.6151	.00198	.00107	.00065
99	1110000110	10.8393	.00193	.00114	.00075
100	1001101001	10.6984	.00187	.00098	.00058
101	1010110001	10.8056	.00177	.00091	.00054
102	1011100001	10.8889	.00174	.00091	.00054
103	1011010001	10.8889	.00170	.00087	.00051
104	1010101001	10.8889	.00166	.00083	.00048
105	1010011001	10.8889	.00166	.00082	.00047
106	1011001001	10.9722	.00160	.00079	.00045
107	1010100101	11.0794	.00148	.00070	.00039
108	1100110001	11.1806	.00146	.00070	.00040
109	1101100001	11.2639	.00143	.00070	.00041
110	1011000101	11.1627	.00141	.00066	.00036
111	1101010001	11.2639	.00140	.00067	.00038
112	1100101001	11.2639	.00137	.00063	.00035
113	1100011001	11.2639	.00136	.00063	.00035
114	1110100001	11.4544	.00132	.00064	.00037
115	1101001001	11.3472	.00131	.00061	.00033
116	1111000001	11.5377	.00130	.00063	.00037
117	1110010001	11.4544	.00129	.00061	.00034
118	1100100101	11.4544	.00121	.00053	.00029

i	$R.O.$	$B_N(Z)$	$P(Z)$ $a=b=2$	$a=b=3$	$a=b=4$
119	1100010101	11.4544	.00121	.00053	.00028
120	1110001001	11.5377	.00121	.00054	.00030
121	1101000101	11.5377	.00116	.00050	.00027
122	1100001101	11.5377	.00115	.00050	.00027
123	1110000101	11.7282	.00106	.00045	.00024
124	1100100011	11.8294	.00099	.00040	.00021
125	1101000011	11.9127	.00095	.00038	.00002
126	1110000011	12.1032	.00087	.00034	.00017

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