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ON A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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§0. Introduction

The writing of this paper has been motivated by recent results of M. S. Robertson [7] and P. M. Chichra [1]. Let $S(\beta,\varphi)$ denote the class of functions

$$F(z)=z+a_2z^2+\cdots\cdots$$

which are regular in $\Delta=\{z: |z|<1\}$ and satisfy the condition

$$\text{(0.1)}\qquad \operatorname{Re}\left\{e^{-i\varphi}\frac{zF'(z)}{F(z)}\right\}\geq\beta\cos\varphi,\ |\varphi|<\frac{\pi}{2},\ 0\leq\beta<1$$

For $\beta=0$, this class was introduced by L. Spacek [9] and called the class of φ-spiral functions. It is composed of the univalent functions and constitutes a subclass of the well known class S. From the definition it results that

$$S(\beta,\varphi)\subset S(0,\varphi)\subset S.$$

For $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, $0\leq\lambda<1$, and some $F(z)\in S(\beta,\varphi)$, we say that $f(z)\in K(\lambda,\alpha,\beta,\varphi)$ provided

(i) $f(z)$ is regular in Δ, $f(0)=0$, $f'(0)=1$

(ii) $f'(z)\neq 0$ in Δ,

(iii) $\operatorname{Re}\left\{e^{-i\alpha}\frac{zf'(z)}{F(z)}\right\}\geq\lambda\cos\alpha,\ z\in\Delta$

We note that the class $K(0,0,0,0)$ constitutes a subclass introduced by Bazilevic [3] of the class of close-to-convex functions with the classical normalization.

In this note we give a useful representation formula for members of $S(\beta,\varphi)$ and we determine the sharp radius of convexity for the functions $f(z)\in K(\lambda,\alpha,\beta,\varphi)$. Finally we establish the sharp upper bound and lower bound of $|f'(z)|$ if $f(z)\in K(\lambda,\alpha,\beta,\varphi)$.

§1. Preliminary remarks

Let $\mathcal{P}$ denote the class of functions $p(z)$ which are regular and satisfy $p(0)=1$, Re $p(z)>0$, for z in Δ. If $F(z)\in S(\beta,\varphi)$ then

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$$\operatorname{Re}\left\{e^{-i\varphi}\frac{zF'(z)}{F(z)}\right\} \geq \beta\cos\varphi,\ 0\leq\beta<1,\ |\varphi|<\frac{\pi}{2}.$$

The introduction of appropriate normalizing factors enables us to write

$$\operatorname{Re}\left(\frac{e^{-i\varphi}\dfrac{zF'(z)}{F(z)}-\beta\cos\varphi+i\sin\varphi}{(1-\beta)\cos\varphi}\right) > 0$$

and

$$\left.\frac{e^{-i\varphi}\dfrac{zF'(z)}{F(z)}-\beta\cos\varphi+\sin\varphi}{(1-\beta)\cos\varphi}\right|_{z=0}=1,\ \text{since}\ \left.\frac{zF'(z)}{F(z)}\right|_{z=0}=1.$$

Hence, we can write

$$\frac{e^{-i\varphi}\dfrac{zF'(z)}{F(z)}-\beta\cos\varphi+i\sin\varphi}{(1-\beta)\cos\varphi}=p(z),\ p(z)\in\mathcal{P}.$$

Thus

$$(1.1)\quad e^{-i\varphi}\frac{zF'(z)}{F(z)}=(1-\beta)\cos\varphi\, p(z)+\beta\cos\varphi-i\sin\varphi.$$

From (1.1) we have

$$\frac{F'(z)}{F(z)}-\frac{1}{z}=(1-\beta)\cos\varphi\, e^{i\varphi}\frac{p(z)-1}{z}$$

By integration, one obtains

$$\log\frac{F(z)}{z}=(1-\beta)\cos\varphi\, e^{i\varphi}\int_0^z\frac{p(t)-1}{t}dt$$

Therefore we have a useful represention formula for members of $S(\beta,\varphi)$ in terms of functions $p(z)$ in $\mathcal{P}$. That is,

$$(1.2)\qquad F(z)=z\exp\left\{(1-\beta)\cos\varphi\, e^{i\varphi}\int_0^z\frac{p(t)-1}{t}dt\right\}.$$

LEMMA 1. *If $F(z)\in S(\beta,\varphi)$, then, for $|z|=r<1$, we have*

$$(1.3)\qquad \left|\frac{zF'(z)}{F(z)}-\frac{1+r^2e^{i\varphi}(e^{i\varphi}-2\beta\cos\varphi)}{1-r^2}\right|\leq\frac{2r(1-\beta)\cos\varphi}{1-r^2}.$$

The equality is attained by the function

$$(1.4)\qquad F(z)=\frac{z}{(1-z)^{2(1-\beta)\cos\varphi\exp(i\varphi)}}.$$

Proof. If $F(z)\in S(\beta,\varphi)$, there exists a function $p(z)\in\mathcal{P}$ such that

(1.5) $$e^{-i\varphi}\frac{zF'(z)}{F(z)}=(1-\beta)\cos\varphi p(z)+\beta\cos\varphi-i\sin\varphi.$$

In profiting of the domain of variation of the function $p(z)\in\mathcal{P}$ for z fixed, we obtain the conclusion of Lemma 1, since that domain of variation is a circle in which the diameter, located on the real axis, has its extremities at the points $\frac{1-r}{1+r}$, $\frac{1+r}{1-r}$.

LEMMA 2. *If* $F(z)\in S(\beta,\varphi)$, *we have, for* $|z|=r<1$,

(1.6) $$\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi-1)}{1-r^2}\leq\operatorname{Re}\frac{zF'(z)}{F(z)}\leq$$

$$\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi+1)}{1-r^2}.$$

The equality is accomplished on the left and on the right respectively, for $|z|=r$, *by the function*

(1.7) $$F_0(z)=\frac{z}{(1-ze^{i\theta})^{2(1-\beta)\cos\varphi\exp(i\varphi)}},\qquad 0\leq\theta\leq2\pi.$$

For the left equality it takes place by setting

$$\theta=2\text{arc }\tan\left(\frac{1-r}{1+r}\cot\frac{\varphi}{2}\right)$$

and for right equality it takes place by setting

$$\theta=2\text{arc }\tan\left[\frac{1-r}{1+r}\cot\left(\frac{\pi}{2}+\frac{\varphi}{2}\right)\right].$$

Proof. From Lemma 1, we have

$$-\frac{2r(1-\beta)\cos\varphi}{1-r^2}+\operatorname{Re}\frac{1+r^2e^{i\varphi}(e^{i\varphi}-2\beta\cos\varphi)}{1-r^2}\leq\operatorname{Re}\frac{zF'(z)}{F(z)}$$

$$\leq\frac{2r(1-\beta)\cos\varphi}{1-r^2}+\operatorname{Re}\frac{1+r^2e^{i\varphi}(e^{i\varphi}-2\beta\cos\varphi)}{1-r^2}.$$

The simple calculation gives the inequality (1.6). We can show that the function

$$F_0(z)=\frac{z}{(1-ze^{i\varphi})^{2(1-\beta)\cos\varphi\exp(i\varphi)}}\in S(\beta,\varphi).$$

Since the equality in (1.6) is real at the point z, $|z|=r$, we have, for $z=r$.

(1.8) $$\operatorname{Re}\frac{rF_0'(r)}{F_0(r)}=1+2(1-\beta)\cos\varphi\frac{r\cos(\varphi+\theta)-r^2\cos\varphi}{1+r^2-2r\cos\varphi}.$$

From Lemma 1, we have

$$(1.9)\qquad \min_{F(z)\in S(\beta,\varphi)} \mathrm{Re}\frac{zF'(z)}{F(z)}=\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi-1)}{1-r^2},\ |z|=r.$$

In (1.8), one is able to choose θ in such a manner that

$$1+2(1-\beta)\cos\varphi\frac{r\cos(\varphi+\theta)-r^2\cos\varphi}{1+r^2-2r\cos\varphi}=\frac{1-r^2+2r(1-\beta)\mathrm{co}\varphi(r\cos\varphi-1)}{1-r^2}$$

holds. After some transformation, we obtain

$$\theta=2\ \mathrm{arc}\ \tan\left(\frac{1-r}{1+r}\cot\frac{\varphi}{2}\right).$$

In order to prove the right equality we have, from Lemma 1,

$$(1.10)\qquad \max_{F(z)\in S(\beta,\varphi)} \mathrm{Re}\frac{zF'(z)}{F(z)}=\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi+1)}{1-r^2},\ |z|=r.$$

Solving the equation

$$1+2(1-\beta)\cos\varphi\frac{r\cos(\varphi+\theta)-r^2\cos\varphi}{1-r^2-2r\cos\varphi}=\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi+1)}{1-r^2,},$$

$|z|=r$, from which one obtains

$$\theta=2\,\mathrm{arc}\tan\left[\frac{1-r}{1+r}\cot\left(\frac{\pi}{2}+\frac{\varphi}{2}\right)\right].$$

LEMMA 3. [5] *Let $p(z)\in\mathcal{P}$, $|z|=r<1$. For each complex number η, $\mathrm{Re}\,\eta<0$, we have*

$$(1.11)\qquad \left|\frac{zp'(z)}{p(z)+\eta}\right|\le\frac{2r}{(1-r)\ [1+r+(1-r)\mathrm{Re}\,\eta]},\ |z|=r.$$

§ 2. Principal results

THEOREM 1. *If $f(z)\in K(\lambda,\alpha,\beta,\varphi)$, the radius of convexity of $f(z)$ is the smallest positive root of the equation*

$$(2.1)\qquad \begin{aligned}&(1-2\lambda)[2(1-\beta)\cos^2\varphi-1]r^3-\{1-2(1-\beta)[\cos^2\varphi-(1-2\lambda)\cos\varphi]\\&+2(1-\lambda)\}r^2-[1+2(1-\beta)\cos\varphi]r+1=0.\end{aligned}$$

The extremal function is of the form

$$(2.2)\qquad f_0(z)=\int_0^z\frac{1-z}{(1-ze^{i\theta})^{2(1-\beta)\cos\varphi\exp(i\varphi)}[1+z(1-2\lambda)]}dz$$

$$\textit{where } \theta = 2 \operatorname{arc}\tan\left(\frac{1-r}{1+r}\cot\frac{\varphi}{2}\right), \quad |z| = r < 1.$$

Proof. If $f(z) \in K(\lambda, \alpha, \beta, \varphi)$, then for some $F(z) \in S(\beta, \varphi)$, $|\alpha| \leq \frac{\pi}{2}$, $0 \leq \lambda < 1$, we have

$$\operatorname{Re}\left[e^{-i\alpha} \frac{zf'(z)}{F(z)}\right] \geq \lambda \cos\alpha, \quad z \in \Delta.$$

Hence,

$$\frac{zF'(z)}{F(z)} = e^{i\alpha}[(1-\lambda)\cos\alpha\, p(z) + \lambda\cos\alpha - i\sin\alpha]. \tag{2.3}$$

Taking logarithm and the derivative with respect to z, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zF'(z)}{F(z)} + \frac{zp'(z)}{p(z)+\eta}, \quad \text{where } \eta = \frac{\lambda}{1-\lambda} - i\frac{\tan\alpha}{1-\lambda}. \tag{2.4}$$

Therefore,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \min_{F(z)\in S(\beta\varphi)} \operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) - \max_{P(z)\in P} \left|\frac{zp'(z)}{p(z)+\eta}\right|. \tag{2.5}$$

But Lemma 2 and Lemma 3 give us that

$$\min_{F(z)\in S(\beta\varphi)} \operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) \geq \frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi-1)}{1-r^2}, \quad |z| = r < 1 \tag{2.6}$$

$$\max_{P(z)\in P} \left|\frac{zp'(z)}{p(z)+\eta}\right| \geq \frac{2r(1-\lambda)}{(1-r)[1+r(1-2\lambda)]} \quad \text{where } \operatorname{Re}\eta = \frac{\lambda}{1-\lambda}. \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.5),

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq$$

$$\frac{(1-2\lambda)[2(1-\beta)\cos^2\varphi-1]r^3 - [1-2(1-\beta)\{\cos^2\varphi-(1-2\lambda)\cos\varphi\}+2(1-\lambda)]r^2}{(1-r^2)[1+r(1-2\lambda)]}$$

$$-\frac{[1+2(1-\beta)\cos\varphi]r-1}{(1-r^2)[1+r(1-2\lambda)]}.$$

If r_0 is the smallest positive root of the equation

$$(1-2\lambda)[2(1-\beta)\cos^2\varphi-1]r^3 - [1-2(1-\beta)\{\cos^2\varphi-(1-2\lambda)\cos\varphi\}+2(1-\lambda)]r^2$$
$$-[1+2(1-\beta)\cos\varphi]r+1=0, \tag{2.8}$$

$f(z) \in K(\lambda, \alpha, \beta, \varphi)$ are convex for z, $|z| \leqq r_0$. To show this result is sharp, it suffices to show that the function

$$(2.10)\qquad f_0(z)=\int_0^z \frac{1-z}{(1-ze^{i\theta})^{2(1-\beta)\cos\varphi\exp(i\varphi)}[1+z(1-2\lambda)]}dz$$

which belongs to the class $K(\lambda,\alpha,\beta,\varphi)$ is not able to be convex in a circle of radius larger than r_0.
From (2.10), we obtain

$$(2.11)\qquad 1+\frac{zf_0''(z)}{f_0'(z)}=\frac{zF_0'(z)}{F_0(z)}-\frac{zp_0'(z)}{p_0(z)+\frac{\lambda}{1-\lambda}}$$

where

$$F_0(z)=\frac{z}{(1-ze^{i\theta})^{2(1-\beta)\cos\varphi\exp(i\varphi)}}\in S(\beta,\varphi),\quad p_0(z)=\frac{1+z}{1-z}.$$

For $z=r$ with $|z|=r<1$, we have

$$(2.12)\qquad \left|\frac{rp_0'(r)}{p_0(r)+\frac{\lambda}{1-\lambda}}\right|=\frac{2r(1-\lambda)}{(1-r)[1+r(1-2\lambda)]}.$$

Taking into accounts of the Lemma 2, and of the formulas (2.11), (1.8), (1.9), we obtain for the function $f_0(z)$

$$\mathrm{Re}\left(1+\frac{rf_0''(r)}{f_0'(r)}\right)=\min_{F(z)\in S(\beta,\varphi)}\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right)-\max_{p(z)\in P}\left|\frac{zp'(z)}{p(z)+\frac{\lambda}{1-\lambda}}\right|$$

i. e.,

$$(2.13)\qquad \mathrm{Re}\left(1+\frac{rf_0''(r)}{f_0'(r)}\right)=\frac{1-r^2+2r(1-\beta)\cos\varphi(r\cos\varphi-1)}{1-r^2}$$

$$-\frac{2r(1-\lambda)}{(1-r)[1+r(1-2\lambda)]}.$$

Consequently, if r_0 is the smallest positive root of the equation (2.1), we have, on the circle $|z|=r_0$,

$$\mathrm{Re}\left(1+\frac{rf_0''(r)}{f_0'(r)}\right)=0 \text{ for } r=r_0.$$

Therefore, r_0 is the radius of convexity for $f(z)\in K(\lambda,\alpha,\beta,\varphi)$ which completes the proof of Theorem 1.

THEOREM 2. *If* $f(z)\in K(\lambda,\alpha,\beta,\varphi)$, *then, for* $|z|=r<1$,

$$(2.14)\qquad \frac{(1-r)^{(1-\beta)(1-\cos\varphi)\cos\varphi+1}}{(1+r)^{(1-\beta)(1+\cos\varphi)\cos\varphi}[1+r(1-2\lambda)]}\le|f'(z)|\le$$

$$\frac{(1-r)^{-(1-\beta)(1+\cos\varphi)\cos\varphi-1}\left[1+r(1-2\lambda)\right]}{(1+r)^{(1-\beta)(1-\cos\varphi)\cos\varphi}}.$$

The equality is attained by the function (2.10).

Proof: Let $f(z)\in K(\lambda,\alpha,\beta,\varphi)$, we can write

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)=1+r\frac{\partial}{\partial r}\log|f'(z)|,\quad z=re^{i\psi},\quad 0\le\psi\le 2\pi,\quad 0\le r\le 1,$$

and

$$(2.15)\quad \min_{f(z)\in K(\lambda,\alpha,\beta,\varphi)}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le \max_{f(z)\in K(\lambda,\alpha,\beta,\varphi)}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)$$

In addition, (2.4) is accompanied directly by the inequalities:

$$\min_{f(z)\in K(\lambda,\alpha,\beta,\varphi)}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\ge\min_{F(z)\in S(\beta,\varphi)}\mathrm{Re}\left(\frac{zF'(z)}{F(z)}\right)-\max_{p(z)\in P}\left|\frac{zp'(z)}{p(z)+\eta}\right|$$

$$\max_{f(z)\in K(\lambda,\alpha,\beta,\varphi)}\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\le\max_{F(z)\in S(\beta,\varphi)}\mathrm{Re}\left(\frac{zE'(z)}{F(z)}\right)+\max_{p(z)\in P}\left|\frac{zp'(z)}{p(z)+\eta}\right|.$$

Hence, taking into account (2.15), (1.10), (1.11) and (1.12) we obtain

$$2(1-\beta)\cos\varphi\frac{r\cos\varphi-1}{1-r^2}-\frac{2(1-\lambda)}{(1-r)[1+r(1-2\lambda)]}\le\frac{\partial}{\partial r}\log|f'(z)|$$
$$\le 2(1-\beta)\cos\varphi\frac{r\cos\varphi+1}{1-r^2}+\frac{2(1-\lambda)}{(1-r)[1+r(1-2\lambda)]}.$$

Integrating in the interval $(0,r)$, we obtain the bound (2.14). The preceeding proof shows directly that (2.10) is the extremal function if we take into account respectively,

$$\theta=2\,\mathrm{arc}\ \tan\left(\frac{1-r}{1+r}\cot\frac{\varphi}{2}\right)$$

and

$$\theta=2\,\mathrm{arc}\ \tan\left[\frac{1-r}{1+r}\cot\left(\frac{\pi}{2}+\frac{\varphi}{2}\right)\right]$$

REMARK. If $\lambda=\beta=0$, let $f(z)\in K(0,\alpha,0,\varphi)$, then $f(z)$ is a convex function in $|z|\le r_0$, where r_0 is the smallest positive root of the equation;

$$(2\cos^2\varphi-1)r^3+(2\cos^2\varphi-2\cos\varphi-3)r^2-(1-2\cos\varphi)r+1=0.$$

We see that

$$r_0=\frac{1+\cos\varphi-\sqrt{1+2\cos\varphi+\sin^2\varphi}}{\cos 2\varphi}.$$

In particular, if $\varphi=0$,

$$r_0=2-\sqrt{3}$$

and the extremal function being the Koebe function

$$f_0(z)=\frac{z}{(1+z)^2}.$$

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