韓國數學敎育學會誌〈數學敎育〉
1977. 12 Vol. XVI No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1977. Vol. XVI No. 1

A Note On Semi-developable Spaces

By Heung Ki Kim

Kang Won National University, Chun Cheon, Korea.

In [5], c-semi-developable space was introduced and it was shown that a space X is c-semi-developable iff X is c-first countable and c-semi-stratifiable. In this note we will show some properties of c-semi-developable space.

Throughout this note, the spaces stand for T_2-spaces and N stands for the natural numbers.

Definition 1. Let $(X, \mathfrak{F})$ be a topological space and let g be a function $g: N \times X \to \mathfrak{F}$. Then g is called a *COC-function* for X if it satisfies these two conditions:

(i) $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all $x \in X$

(ii) $g(n+1, x) \subset g(n, x)$ for all $n \in N$ and $x \in X$.

Definition 2. A space X is *c-semi-stratifiable* if there is a COC-function which satisfies the condition: If A is a closed compact subset of X and $x \in X-A$, then there exists n such that $x \notin g(n, a)$ for each $a \in A$.

Lemma 1. *The following statements are equivalent:*

(1) *X is c-semi-stratifiable.*

(2) *There is a COC-function such that if $x \in g(n, x_n)$ for all $n \in N$ and $x \in U$ for a cocompact open set U, then $\langle x_n \rangle$ is eventually in U.*

(3) *To each cocompact open set U, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that*

(i) $U = \bigcup_{n=1}^{\infty} U_n$

(ii) $U_{n+1} \supset U_n$

(iii) *$U_n \subset V_n$ whenever $U \subset V$, where $\{V_n\}_{n=1}^{\infty}$ is the sequence assigned to cocompact open set V.*

(4) *To each compact closed set A, one can assign a sequence $\{A_n\}_{n=1}^{\infty}$ of open subsets of X such that*

(i) $A = \bigcap_{n=1}^{\infty} A_n$

(ii) $A_{n+1} \subset A_n$

(iii) $A_n \subset B_n$ *whenever* $A \subset B$, *where* $\{B\}_{n=1}^{\infty}$ *is the sequence assigned to compact closed set* B.

Proof. (1)⇒(2). Let $x \in g(n, x_n)$ for all $n \in \mathrm{N}$ and $x \in \mathrm{U}$. Then $x \notin \mathrm{A} = \mathrm{X} - \mathrm{U}$. Since $\mathrm{X} - \mathrm{U} = \mathrm{A}$ is closed and compact, from the hypothesis, there exists n_0 such that $x \notin g(n_0, \mathrm{A})$, and also if $n \geqq n_0$, then $x \notin g(n_0, \mathrm{A}) \supset g(n, \mathrm{A})$. If $x_n \in \mathrm{A}$ for $n \geqq n_0$, then $x \in g(n, x_n) \subset g(n, \mathrm{A})$. This contradicts to $x \notin g(n, \mathrm{A})$. Hence $x_n \notin \mathrm{A} = \mathrm{X} - \mathrm{U}$ for $n \geqq n_0$, that is, $x_n \in \mathrm{U}$ for $n \geqq n_0$.
(2)⇒(3). Let $\mathrm{U}_n = \mathrm{X} - g(n, \mathrm{X} - \mathrm{U}) = \mathrm{X} - \bigcup_{x \in \mathrm{X} - \mathrm{U}} g(n, x)$ for a cocompact open set U and for all $n \in \mathrm{N}$. Then U_n is a closed subset of X. Have to show that (i), (ii) and (iii) in (3) are satisfied.

(i) $x \notin \bigcup_{n=1}^{\infty} \mathrm{U}_n \Rightarrow x \notin \mathrm{U}_n = \mathrm{X} - g(n, \mathrm{X} - \mathrm{U})$ for all n

$\Rightarrow x \in g(n, \mathrm{X} - \mathrm{U})$ for all n

$\Rightarrow x \in \bigcup_{x_n \in \mathrm{X} - \mathrm{U}} g(n, x_n)$ for all n

$\Rightarrow$ there exists $x_n \in \mathrm{X} - \mathrm{U}$ such that $x \in g(n, x_n)$ for all n

$\Rightarrow x \notin \mathrm{U}$ [$\because$ $x \in \mathrm{U}$ and $x \in g(n, x_n) \Rightarrow \exists n_0$ such that $x_n \in \mathrm{U}$ for all $n \geqq n_0$].

Therefore $\mathrm{U} \subset \bigcup_{n=1}^{\infty} \mathrm{U}_n$. Since $\mathrm{U}_n = \mathrm{X} - g(n, \mathrm{X} - \mathrm{U}) = \mathrm{X} - \bigcup_{x \in \mathrm{X} - \mathrm{U}} g(n, \mathrm{X}) \subset \mathrm{X} - (\mathrm{X} - \mathrm{U}) = \mathrm{U}$ for all n, $\bigcup_{n=1}^{\infty} \mathrm{U}_n \subset \mathrm{U}$.

(ii) $\mathrm{U}_n = \mathrm{X} - g(n, \mathrm{X} - \mathrm{U}) = \mathrm{X} - \bigcup_{n \in \mathrm{X} - \mathrm{U}} g(n, x) \subset \mathrm{X} - \bigcup_{x \in \mathrm{X} - \mathrm{U}} g(n+1, x) = \mathrm{U}_{n+1}$.

(iii) If $\mathrm{U} \subset \mathrm{V}$, then $g(n, \mathrm{X} - \mathrm{U}) \supset g(n, \mathrm{X} - \mathrm{V})$. Hence $\mathrm{U}_n = \mathrm{X} - g(n, \mathrm{X} - \mathrm{U}) \subset \mathrm{X} - g(\mathrm{N}, \mathrm{X} - \mathrm{V}) = \mathrm{V}_n$.

(3)⇒(1). Since $\mathrm{X} - \{x\}$ is a cocompact open set for each $x \in \mathrm{X}$, there is a sequence $\{(\mathrm{X} - \{x\})_n : n \in \mathrm{N}\}$ of closed subsets of X, which satisfies (3).
Let $g(n, x) = \mathrm{X} - \{(\mathrm{X} - \{x\})_n : n \in N\}$. Then

(a) $g(n, x) = \mathrm{X} - \{(\mathrm{X} - \{x\})_n : n \in \mathrm{N}\} \supset \mathrm{X} - (\mathrm{X} - \{x\}) = \{x\} \ni x$ for all n.

(b) Since $(\mathrm{X} - \{x\})_n \subset (\mathrm{X} - \{x\})_{n+1}$, $g(n+1, x) = \mathrm{X} - [(\mathrm{X} - \{x\})_{n+1}] \subset \mathrm{X} - [(\mathrm{X} - \{x\})_n] = g(n, x)$.

(c) Let A be a compact closed set and $x \in \mathrm{X} - \mathrm{A}$. Since $\bigcup_{n=1}^{\infty} (\mathrm{X} - \mathrm{A})_n = \mathrm{X} - \mathrm{A} \ni x$,
$g(n, \mathrm{A}) = \bigcup_{a \in \mathrm{A}} g(n, a) = \bigcup_{a \in \mathrm{A}} [\mathrm{X} - (\mathrm{X} - \{a\})_n] = \mathrm{X} - \bigcap_{a \in \mathrm{A}} [(\mathrm{X} - \{a\})_n]$ and $(\mathrm{X} - \mathrm{A})_n \subset (\mathrm{X} - \{a\})_n$, there exists an n_0 such that $x \notin \mathrm{X} - (\mathrm{X} - \mathrm{A})_{n_0} \supset \mathrm{X} - (\mathrm{X} - \{a\})_{n_0} = g(n_0, a)$ for all $a \in \mathrm{A}$.
(3)⇔(4). Clear.

Theorem 2. *The countable product of c-semi-stratifiable spaces is c-semi-stratifiable.*
Proof. For each i, let X_i be a c-semi-stratifiable space and $\{g_{ij}\}_{j=1}^{\infty}$ be a sequence of functions on X_i satisfying the condition (2) of Lemma 1. Then it can be proved by the same method of Theorem 2.1 in [3].

Definition 3. A space X is *c-first countable* iff $\mathrm{K} \cap \{x\} = \phi$ for a closed compact subset K then there exists n such that $\mathrm{K} \cap g(n, x) = \phi$ or, if V is the complement of closed and compact set and $x \in \mathrm{V}$, then there exists n such that $g(n, x) \subset \mathrm{V}$.

Theorem 3. *The countable product of c-first countable space is c-first countable.*
Proof. Let $\{g_{ij}\}_{j=1}^{\infty}$ be a sequence of functions on X_i which satisfies definition 3, and let $h_{ij}(x)=\begin{cases} g_{ij}(\pi_i(x)), & j\leq i \\ X_i & j>i \end{cases}$ where $\pi_i:\prod_i X_i \to X_i$ is the projection. Put $g_j(x)=\prod_{i=1}^{\infty} h_{ij}(x)$, then $x\in \cap_j g_j(x)$ and $g_{j+1}(x)\subset g_j(x)$. Also, if $K\cap\{x\}=\phi$ for a closed compact set K then there exists n such that $g_n(x)\cap K=\phi$.

Definition4. A space X is *c-semi-developable* iff there is a sequence $\{\gamma_1, \gamma_2, \cdots\}$ of cover of X such that $x\in st(x,\gamma_n)^0$ for each $x\in X$ and if $x\in U$ for some cocompact open set U then there exists n such that $st(x,\gamma_n)\subset U$.

Corollary 4. *The countable product of c-semi-developable space is c-semi-developable.*
Proof. By [5], X is c-semi-developable space iff X is c-semi-stratifiable and c-first countable space. Hence this is clear from Theorem 2 and Theorem 3.

Theorem. 5. *A c-semi-stratifiable space X is hereditarily c-semi-stratifiable,*
Proof. Let A be a compact closed subset of Y, then A is compact closedset of X. By Lemma 1, there exists a sequence $\{A_n\}$ of open subsets of X such that
(1) $A=\cap A_n$ (2) $A_{n+1}\subset A_n$ (3) $A_n\subset B_n$ whenever $A\subset B$.
Put $A_n'=A_n\cap Y$, then A_n' is open set in Y and $\cap_n A_n'=\cap_n(A_n\cap Y)=(\cap_n A_n)\cap Y=A\cap Y=A$. Also $A'_{n+1}=A_{n+1}\cap Y\subset A_n\cap Y=A_n'$. If $A\subset B$, where A and B are compact closed subsets of Y, there exist sequences $\{A_n\}$ and $\{B_n\}$ of open subsets of X such that $A_n\subset B_n$. But $A_n'=A_n\cap Y$ and $B_n'=B_n\cap Y$. Therefore $A_n'\subset B_n'$.

Theorem 6. *A c-first countable space X is hereditarily c-first countable.*
Proof. Let K be a compact closed subset of Y. Then $K=Y\cap K'$ for some compact closed subset K' of X. Hence, $\{x\}\cap K=\phi \Rightarrow \{x\}\cap(Y\cap K')=\phi \Rightarrow \{x\}\cap K'=\phi \Rightarrow \exists n$ such that $\phi=g(n,x)\cap K'\supset g(n,x)\cap K$.

Corollary 7. *A c-semi-developable space is hereditarily c-semi-developable.*
Proof. By Theorem 5 and Theorem 6, this theorem is clear.

REFERENCES

1. Charles C. Alexender, (1971), Semi-developable space and quotient images of metric spaces, *Pacific J. Math.*, Vol. 37, 277-293.
2. C. Borges, (1966), On stratifiable spaces, *Pacific J. Math.*, Vol. 17, 1-16.
3. Geoffrey D. Creede, (1970), Concerning semi-stratifiable spaces, *Pacific J. Math.*, Vol. 32, 47-54.
4. Harold W. Martin, (1973), Metrizability of M-spaces, *Canad. J. Math.*, Vol. 25, 840-841.
5. H.K. Kim, (1976), A note on c-semi-developable spaces and c-first countable spaces, *J. of Korea Math. Ed.*, Vol. 15, 47-49.
6. J. Kelly (1955), *General topology*, Van Nostrand