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A Collection of Inequivalently Imbedded Arcs in n-Space

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0. Introduction

The purpose of this thesis is to give an explicit construction of uncountably many arcs in the n-dimensional space, $n>3$, any two of which are not equivalently imbedded.

By suspending S^{n-1} modulo an arc, we make three distinctly imbedded arcs in S^n. Then we paste countably many copies of these arcs together and obtain many examples of arcs in the n-space. The construction procedure follows the routine method like that of [6]. However, we must be carefull in joining the building blocks because the geometrical vision is not so clear as in the 3-dimensional case.

1. Terminologies and Notaions

The unit n-simplex and the unit n-sphere are denoted by s^n and S^n. Homeomorphic images of s^n or S^n are called n-cells or n-spheres, respectively. Two subsets A and B of a space S are called *equivalently imbedded* if they are homeomorphic and there is a pair homeomorphism from the pair (S, A) to (S, B), For $x\in A$ and $y\in B$, we say that A is equivalently imbedded at x as B is at y if there is a neighborhood U of x and a pair homeomorphism h from $(S, A\cap U)$ to $(S, B\cap h(U))$ with $h(x)=y$.

If A is a k-cell or k-sphere in an n-sphere S, and if there is a pair homeomorphism of (S, A) to (S^n, B) with B a subpolyhedron of S^n, then we say that A is *tame* in S. If A is an $(n-1)$-sphere and (S, A) is pair homeomorphic with (S^n, S^{n-1}), then we say A is *flat* in S. The notion of being locally tame or locally flat will then have obvious meaning.

Let A be a closed subset of a space X and let x be a limit point of $X-A$ lying in A such that every neighborhood of x contains a neighborhood of x whose intersection with $X-A$ is arcwise connected. We say that $X-A$ is *locally simply connected at x* provided that if U is a neighborhood of x then there is a neighborhood V of x such that every simple closed curve in $V-A$ is null homotopic in $U-A$. Trivially, the complement of a tame arc in the n-space, $n>3$, is locally simply connected at each point of the arc, and if there is a pair homeomorphism h from (X, A) to (Y, B) and $X-A$ is locally simply connected at x then $Y-B$ is so at h(x).

The suspension S(X) of a compact space X is the space obtained from $X\times I$ by collapsing down the compact sets $X\times 0$ and $X\times 1$ to points. The space X/σ is the quotient space of X by identifying an arc σ in X to a point.

2. The three arcs

Proposition A (Andrews–Curtis). *If σ is an arc in S^{n-1} and D is a compact neighborhood of σ, then $S(D/\sigma)$ is topologically $S(D)$. In particular, $S(S^{n-1}/\sigma)$ is an n-sphere.*

Proof. By [1], $D/\sigma\times R^1$ and $D\times R^1$ are homeomorphic. Hence the proposition follows by taking the two point compactifications of $D/\sigma\times R^1$.

By Fox and Artin [6], there is an arc σ in S^3 such that $S^3-\sigma$ is not simply connected. Then the arc in the 4-sphere $S(S^3/\sigma)$ joining the suspension vertices through σ has non-simply connected complement as will be seen in Lemma 1 below. Since this type of induction works further, we obtain the following result which is originally proved by direct constructions in the n-space, $n\geq 3$ [2].

Proposition B (Antoine–Blankinship). *For each $n\geq 3$, S^n contains an arc whose complement is not simply connected.*

Now suppose $n\geq 4$ and let σ be an arc in S^{n-1} contained in $S^{n-1}-S^{n-2}$ such that $S^{n-1}-\sigma$ is not simply connected. Throughout this paper, α will denote the arc in $S(S^{n-1}/\sigma)$ joining the suspension vertices through σ, that is, α is the part of $S(S^{n-1}/\sigma)$ corresponding to the subset $\sigma\times I$ of the product $S^{n-1}/\sigma\times I$. Let β be the subarc of β corresponding to $\sigma\times[0,\frac{1}{2}]$, and γ be the subarc of β corresponding to $\sigma\times[\frac{1}{4},\frac{1}{2}]$.

Lemma 1. *The arcs α, β and γ satisfy the followings*

(*a*) *The complement of α is not simply connected.*

(*b*) *The complement of β is simply connected.*

(*c*) *The complement of γ is simply connected.*

Proof. (a) Since $S(S^{n-1}/\sigma)-\alpha=(S^{n-1}-\sigma)\times(0,1)$ deformation retracts to $(S^{n-1}-\sigma)\times 0$, $S(S^{n-1}/\sigma)-\alpha$ has the same homotopy type as $S^{n-1}-\sigma$. This proves (a) since $S^{n-1}-\sigma$ is not simply connected.

(b) $S(S^{n-1}/\sigma)-\beta$ deformation retracts to the vertex $(\sigma,1)$ by the map sending$((x,t),s)$ to $(x,s(1-t)+t)$.

(c) Let $f:S^1\longrightarrow S(S^{n-1}/\sigma)-\gamma$ be a loop then there is a positive number ε and a neighborhood U of σ in S^{n-1}/σ such that $f(S^1)$ does not meet $U\times(\frac{1}{2}-\varepsilon,\ \frac{1}{4}+\varepsilon)$. By [3], there is a homeomorphism h sending $S^{n-1}\times(0,1)$ onto $S^{n-1}/\sigma\times(\frac{1}{4}-\varepsilon,\ \frac{1}{2})$. Let V be the component of $S(S^{n-1}/\sigma)-h(S^{n-1}\times\frac{1}{2})$ containing $(\sigma,0)$. Then the closure D of V is an n-cell by [3] and the distance δ between $D-V$ and $S^{n-1}/\sigma\times[0,\ \frac{1}{2}-\varepsilon]$ is positive.

Now if $f^{-1}(V)=\phi$ or $f^{-1}(V)=S^1$, f is null homotopic in $S(S^{n-1}/\sigma)-(V\cup\gamma)$ or in V. Otherwise, $f^{-1}(V)=\cup\mathcal{a}$ where $\mathcal{a}$ is a collection of disjoint open arcs in S^1, and by the uniform continuity of f, $\mathcal{a}$ satisfies:

Diam $(f(A))>\sigma$ for only finite $A_1, A_2, \cdots, A_k$ of $\mathcal{a}$.

Since β cannot disconnect the $(n-1)$-sphere $D-V$. for each A_i, there is a continuous map $g_i:A_i \longrightarrow (D-V)-\beta$ such that g_i agrees with f on $\bar{A}_i - A_i$. And D is an n-cell and each f A_i is homotopic with g_i relative to $A_i - A_i$, so there is a continuous map $g:S^1 \longrightarrow S(S^{n-1}/\sigma)-\beta$ with g homotopic to f in $S(S^{n-1}/\sigma)-\gamma$. By (b), g is null homotopic in $S(S^{n-1}/\sigma)-\beta$ and so is f in $S(S^{n-1}/\sigma)-\gamma$. This completes the proof.

Lemma 2. *The complement of β is not locally simply connected at an end point of β.*

Proof. Let $U=S^{n-1}/\sigma\times[0, \frac{1}{2})$ be a neighborhood of $(\sigma, 0)$, then any neighborhood W of $(\sigma, 0)$ in U contains a neighborhood V with $V-\beta=(S^{n-1}-\sigma)\times(0, \varepsilon)$, $\varepsilon>0$. Since the inclusions $V-\beta\subset U-\beta\subset S(S^{n-1}/\sigma)-\alpha$ are clearly homotopic, by lemma 1, $V-\beta$ contains a loop which is not null homotopic in $U-\beta$. Hence $W-\beta$ contains a loop not null homotopic in $U-\beta$.

Lemma 3. *The complement of γ is locally simply connected at each endpoint of γ.*

Proof. If U is a neighborhood of $(\sigma, \frac{1}{2})$ in $S(S^{n-1}/\sigma)$, there is a neighborhood W of σ in S^{n-1}/σ and an ε such that $V=W\times[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]\subset U$, $0<\varepsilon<\frac{1}{4}$. Here we may suppose W is small enough so that an n-cell D containing $W\times(\frac{1}{2}+\varepsilon)$ is contained in $U-\gamma$, $V-\gamma$ deformation retracts to $W\times(\frac{1}{2}+\varepsilon)$, and eacth loop in $V-\gamma$ is null homotopic in $(V-\gamma)\cup D\subset U-\gamma$. Thus $S(S^{n-1}/\sigma)-\gamma$ is locally simply connected at $(\sigma, \frac{1}{2})$. Similarly, it is locally simply connected at the other end point.

3. The Main Result

Since the arc σ is contained in $s^{n-1}-S^{n-2}$, it follows from Proposition A that $S(s^{n-1}/\sigma)$ is an n-cell and the boundary $S(S^{n-2})$ is a flat $(n-1)$-sphere in $S(S^{n-1}/\sigma)$ whose intersection with α is the two end points of α. Let $R^n\cup\infty$ be the one point compactification of R^n, h be a pair homeomorphism of $(S(S^{n-1}/\sigma), S(s^{n-1}/\sigma))$ to $(R^n\cup\infty, I^n)$ and let $p=h((\sigma,1))$, $q=h((\sigma,\frac{1}{2}))$. Since the arc $h(\beta)$ cannot disconnect I^n, there is a polygonal arc K_1 from p to p_1 such that

(i) K_1 lies in $I^n-h(\beta)$, and

(ii) p_1 lies in the 2^{-1}-neighborhood of q.

Similarly, there is a polygonal arc K_2 from $p_1$to p_2 such that

(i) K_2-p_1 lies in $I^n-h(\beta)\cup K_1$, and

(ii) p_2 lies in the 2^{-2}-neighborhood of q.

Continuing, we obtain an arc K from p to q which is the union of $K_1, K_2, \cdots$, and q. By Cantrell and Edwards [5], the almost polygonal arc K is tame and there is a self homeomorphism g of $R^n\cup\infty$ sending K to a polygonal arc. Let F be a continuous map of $R^n\cup\infty$ onto itself such that the restriction of F on $R^n\cup\infty-g(K)$ is a homeomorphism, $Fg(K)$ is a point and F is the identity on $gh(\beta)$.

The image under this map F of the flat $(n-1)$-sphere $gh(S(S^{n-2}))$ is a topological $(n-1)$-sphere which is locally flat possibly except at $Fgh((\sigma, \frac{1}{2}))$. Therefore, it is flat by Cantrell [4] and $Fgh(S(s^{n-1}/\sigma))$ is an n-cell [3]. Thus $E=h^{-1}g^{-1}Fgh(S(s^{n-1}/\sigma))$ is n-cell containing β such that the boundary of E meets β at the and points of β.

Hence we have proved the following;

Propostion C. *There is an n-cell* E *containing* β *such that the boundary of* E *meets* β *at the end points of* β.

Similarly, there is an n-cell D containing α such that the boundary of D meets α at the end points of α.

Now let D_k, $k=1, 2\cdots$, denotes the set of those points $x=(x_1, x_2 \cdots, x_n)$ in the unit n-simplex s^n with $1-2^{-(k-1)}\leq x_n \leq 1-2^{-k}$. And let f_k denote the obvious linear homeomorphism of s^n to that part of s^n contained in the half space $1-2^{-(k-1)}\leq x_n$ which projects the base simplex s^{n-1} into the hyperplane $x_n=1-2^{-(k-1)}$ towards the vertex $e_n=(0, 0, \cdots, 0, 1)$. Then f_k sends D_1 onto D_k. Let v_1 be the barrycenter of s^{n-1}, and let $v_k=f_k(v_1)$, $k\geq 2$.

Using Proposition C, let δ_0 and δ_1 be ares joining v_1 to v_2 in D_1 such that $(D_1. \delta_0)$ and $(D_1. \delta_1)$ are pair homeomorphic with (D, α) and (E, β), respectively. Now let C be the set of all $\{0, 1\}$ -valued sequences.

For an element c of C. we construct an are δ_c as follows; for each k, let $\delta_{c,k}$ be the are f_k (δ_0) or f_k (δ_1) according to as c(k) is 0 or 1. Then δ_c is the union of $\delta_{c,1}$, $\delta_{c,2}, \cdots$, and the vertex e_n.

Theorem. *If c and c′ are distinct members of* C, *then* δ_c *and* δ_c' *are not equivalently imbedded in* S^n.

Proof. Since $\delta_{c,k}$ is imbedded in S^n as α or β is in $S(S^{n-1}/\sigma)$, the subarcs of δ_c joining v_k to e_n has complement which is not locally simply connected at v_k by lemma 2. On the other hand, if x is a point of δ_c not equal to any of the points e_n, v_1, $v_2, \cdots$, then by lemma 3 the subare of δ_c from x to e_n has complement which is locally simply connected at x.

Hence if there is a pair homeomorphism H of (s^n, δ_c) to $(S^n, \delta_{c'})$ then H must send the set $Z=\{e_n, v_1, v_2, \cdots\}$ onto Z. Put, since any self homeomorphism of the interval I is either order preserving or order reversing, this implies that H sends each v_k to v_k as e_n is the only limit point of Z. Thus existence of a pair homeomorphism between(S^n, δ_c) and $(S^n, \delta_{c'})$ implies that each $\delta_{c,k}$ is equivalently imbedded with $\delta_{c',k}$ in S^n, However, if this is the case, then c=c, because α and β are not equivalently imbedded in $S(S^{n-1}/\sigma)$ by lemma 1. This completes the proof.

Note finally that, since C is equipotent with reals, we have constructed an uncountably infinite collection of inequivalently imbedded ares in n-space.

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