韓國數學教育學會誌〈數學教育〉
1977 12. Vol. XVI No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1977. Vol. XVI No. 1

On The Category Of G-sets

By Im Geun Bin

Cheong Ju University, Cheong Ju, Korea.

1. Introduction.

Let G be a group and S a set. An operation of G on S is a map $f: G\times S \longrightarrow S$ which, if we denote $f(g, s)$ by gs for all g in G and s in S, satisfies;

(i) $1s=s$ for all s in S and the identity element $1\in G$,

(ii) $(g_1g_2)s=g_1(g_2s)$ for all $g_1, g_2\in G$ ([1], [2]).

A set S together with an operation of G on S is called a *G-set* ([1]).

The category $\mathscr{G}(G)$ whose object is the single group G and set of morphisms $\mathrm{Hom}_{\mathscr{G}(G)}(G, G)$ is the set G is called the category of the group G.

Definition. Let G be a group. If S_1 and S_2 are G-sets, then a G-morphism from S_1 to S_2 is a map of the sets $f: S_1\rightarrow S_2$ satisfying

$$f(gs)=gf(s) \text{ for all } g \text{ in G and s in S.}$$

The category with objects the G-sets and morphisms the G-morphisms is called the *category of G-sets* and denoted by G-sets.

In this paper we shall prove the theorem: The category Funct($\mathscr{G}$(G), Sects) of functors from the category of the group G to the category Sets of all sets is equivalent to the category G-sets.

2. G-sets.

Proposition. (a) *For each G-set S, $id_s: S\longrightarrow S$ is a G-morphism.*

(b) *If S_1, S_2, S_3 are G-sets and $f: S_1\longrightarrow S_2$ and $h: S_2\longrightarrow S_3$ are G-morphisms, then the ordinary composition of maps $hf: S_1\longrightarrow S_3$ is a G-morphism.*

(c) *The following data define a category which is called the category of G-sets and is denoted by G-Sets.*

(i) *The objects of G-Sets are the G-sets.*

(ii) *For each pair of objects S_1 and S_2 of G-Sets, G-Sets(S_1, S_2) is the set of all G-morphisms from S_1 to S_2.*

(iii) *For each triple S_1, S_2 and S_3 of objects of G-Sets, the composition map*

G-sets$(S_1, S_2)\times$G-Sets$(S_2, S_3)\longrightarrow$G-Sets$(S_1, S_3)$ is given by $(f, g)\longrightarrow gf$, the ordinary composition of maps.

Proof. (a) $id_s(gs)=gs=gid_s(s)$ for all g in G and s in S.

(b) $hf(gs)=h(gf(s))=g(h(f(s)))=g(hf(s))$ for all g in G and s in S_1.

(c) If S_1, S_2, S_3, S_4 are objects in G-sets and f is in G-sets (S_1, S_2), g is in G-sects (S_2, S_3), and h is in G-sets (S_3, S_4), then $h(gf)=(hg)f$ since the ordinary composition of maps is associative and $h(gf)$ and $(hg)f$ are G-morphisms by (a).

Next for each object S in G-sets, there is an id_s in G-Sets (S, S) such that for each object S_1 in G-sets, we have $fid_s=f$ for all f in G-sets (S, S_1) while $id_s g=g$ for all g in G-sets (S_1, S), which completes the proof.

3. The category G-sets.

By the proposition in 2, we obtain the category G-sets with objects G-sets and arrow G-morphisms.

Lemma 1. *Let G be a group and $\mathscr{C}(G)$ be the category of G. Let $F: \mathscr{C}(G) \longrightarrow Sets$ be a functor. If $\alpha(F)$ is the set $F(G)$ and for each g in G and s in $F(G)$, $f(g, s)=gs =F(g)(s)$, then f is an operation of G on $F(G)$ and $\alpha(F)$ is the G-set.*

Proof. For the identity element $1 \in G$ and for $g_1, g_2 \in G$, $1s=F(1)(s)=1_{F(G)}(s)=s$ for all s in F(G).

$(g_1, g_2)s=F(g_1g_2)(s)=F(g_1)F(g_2)(s)=F(g_1)\ (F(g_2)(s))=g_1(g_2s)$.

Lemma 2. *Let $F_1, F_2: \mathscr{C}(G) \longrightarrow Sets$ be functors and let $\rho: F_1 \longrightarrow F_2$ be a morphism of functors. Then the map*

$$\rho_G : \underset{\displaystyle F_1(G)}{\underset{\|}{\alpha(F_1)}} \longrightarrow \underset{\displaystyle F_2(G)}{\underset{\|}{\alpha(F_2)}}$$

is a morphism of G-sets.

Proof. For all g in G and s in $\alpha(F_1)$, $\rho_G(gs)=\rho_G(F_1(g)(s))=F_2(g)(\rho_G(s))=g(\rho_G(s))$, since ρ is morphism of functors.

Theorem. *Let Funct($\mathscr{C}(G)$, Sets) be the category of functors from the category of the group G to the category Sets of all sets. For every functor $F: \mathscr{C}(G) \longrightarrow Sets$ and for all morphisms of functors $\rho: F_1 \longrightarrow F_2$ in Funct $\mathscr{C}(G)$, Sets), if $\alpha(F)=F(G)$ and $\alpha(\rho)=\rho_G$: $F_1(G) \longrightarrow F_2(G)$, then α is an equivalence functor from the category Funct($\mathscr{C}(G)$, Set(s) to the category G-sets.*

Proof. For two morphisms $\rho_1: F_1 \longrightarrow F_2$ and $\rho_2: F_2 \longrightarrow F_3$ in Funct ($\mathscr{C}(G)$, Sets),

$$\alpha(\rho_2\rho_1)=(\rho_2\rho_1)_G=(\rho_2)_G(\rho_1)_G=\alpha(\rho_2)\alpha(\rho_1)$$

and for the identity morphism $1_F: F \longrightarrow F$,

$$\alpha(1_F)=(1_F)_G=1_{F(G)}=1_{\alpha(F)}$$

Therefore α is a functor.

Next, for every G-sets S in Ob(G-sets) and all G-morphisms $f: S_1 \longrightarrow S_2$ in G-sets, if $\beta(S)(G)=S \in Ob(Sets)$ and $\beta(f)_G=f$, then for $S_1 \xrightarrow{f_1} S_2 \xrightarrow{f_2} S_3$ in G-sets

$$(\beta(f_2f_1))_G=f_2f_1=\beta(f_2)_G\beta(f_1)_G=(\beta(f_2)\beta(f_1))_G$$

and for $1_S: S \longrightarrow S$, $\beta(1_S)_G=1_S=1_{\beta(S)(G)}$

Hence β is also a functor from G-sets to Funct ($\mathcal{O}$(G), Sets).

$$(\beta\alpha(F)) = \beta(\alpha(F)) = \beta(F(G)) \in \text{Funct } (\mathcal{O}(G), \text{ Sets}).$$

But $(\beta\alpha(F))(G) = \beta(FG))(G) = F(G) = [1_{\text{Fund}(\mathcal{O}(G),\text{Sets})}(F)](G)$

Therefore $\beta\alpha = 1_{\text{Funct}(\mathcal{O}(G),\ \text{Sets})}$

Similarly $\alpha\beta = 1_{S\text{-sets}}$, which completes the proof.

REFERENCES

1. M. Auslander, D. A. Buchsbaum, (1974) *Group, Rings, Modules.* Harper & Raw New York.
2. N. Jacobson, (1974) *Basic Algebra* 1. W. H. Freeman and Co. San Francisco.