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On semi-stratifiable (mod K) Spaces

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1. Introduction

H.R. Bennett and H.W. Martin[2] introduced the notions of various types of bases (mod K), especially first-countable(mod K) spaces and developable(mod K) spaces, motivated by Arhangel'skiĭ[1. Theorem 22], Michael and Lutzer[7]. In [3], semi-developable (mod K) spaces are added to these two spaces. It is reasonable to consider semi-stratifiable (mod K) spaces in comparison with the spaces having usual basis.

The main results of this note are;

(1) A semi-stratifiable(mod K) T_1-space is a β-space.

(2) A regular T_1-space is semi-stratifiable if and only if it is c-semistratifiable and semi-stratifiable(mod K).

Throught this note, a COC-map (countable open covering map) for a topological space X means a map g on $N\times X$ to a topology of X with $x\in g(n,x)$ and $g(n+1,x)\subset g(n,x)$ for each $n\in N$ and $x\in X$.

2. Basic concepts

Definition 2.1. A topological space X is *semi-stratifiable*[4] if, to each open set $U\subset X$, one can assign a sequence $\{U_n\}_{n=1}^{\infty}$ of closed subsets of X such that

(1) $\bigcup_{n=1}^{\infty} U_n=U$,

(2) $U_n\subset V_n$ whenever $U\subset V$, where $\{V_n\}_{n=1}^{\infty}$ is a sequence assigned to an open set V.

Creede[4] has shown that a necessary and sufficient condition for a topological space X to be semi-stratifiable is that there exists a COC-map g for X such that if x is a point of X and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X with $x\in g(n, x_n)$ for each $n\in N$, then $\{x_n\}_{n=1}^{\infty}$ converges to x.

Definition 2.2. A topological space X is *c-semi-stratifiable*[6], if there exists a COC-map g for X such that if A is a closed compact subset of X and $x\in X-A$, then there is a natural number n such that $x\notin g(n,a)$ for all $a\in A$.

Definition 2.3. A topological space X is a *β-space*[5] if there is a COC-map g for X such that if x is a point in X and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X with $x\in g(n,x_n)$

for all $n \in N$, then $\{x_n\}_{n=1}^{\infty}$ has a cluster point.

It is well known that every semi-stratifiable space is a c-semi-stratifiable β-space.

3, Semi-stratfiable(mod K) space.

From now on we will use the letter U to denote an open subset of a topological space.

Definition 3.1. Let X be a topological space and $\mathcal{K}$ be a compact covering of X(a covering by compact subsets of X). A covering $\mathcal{g}$ of X is called a *basis*(mod K) [2] if

(1) $\mathcal{g}$ is an open covering,

(2) whenever $x \in K \in \mathcal{K}$ and $K \subset U$, there exists an open set G in $\mathcal{g}$ such that $x \in G \subset U$.

In this case, X is written as the ordered triple $X = (X, \mathcal{K}, \mathcal{g})$.

Definition 3.2. A topological space $X = (X, \mathcal{K}, \mathcal{g})$ is *first-countable*(mod K) [2] if $\mathcal{g}$ is an open covering of X such that: For each point $x \in X$, there exists a sequence $\{G_n(x) \mid n \in N\}$ in $\mathcal{g}$ such that if $x \in K \in \mathcal{K}$ and $K \subset U$, then $x \in G_n(x) \subset U$ for some n.

Definition 3.3. A topological space $X = (X, \mathcal{K}, \mathcal{g})$ is *semi-developable* (mod K) [3] if $\mathcal{g} = \cup \{\mathcal{g}_n \mid n \in N\}$ where $\mathcal{g}_n$ is a covering of X and $\mathcal{g}_{n+1} \propto \mathcal{g}_n$ for all n such that

(1) $x \in st(x.\ \mathcal{g}_n)^\circ$ for all $n \in N$,

(2) if $x \in K \in \mathcal{K}$ and $K \subset U$, then $st(x,\ \mathcal{g}_n) \subset U$ for some n.

We will consider another space with bases(mod K), which was introduced but unnamed [3. Lemma 4.4].

Definition 3.4. A topological space $X = (X, \mathcal{K}, \mathcal{g})$ is *semi-stratifiable* (mod K) if $\mathcal{g}$ is an open covering of X and there exists a COC-map g for X such that if $x \in K \in \mathcal{K}$, $K \subset U$ and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X with $x \in g(n, x_n)$ for each n, then $\{x_n\}_{n=1}^{\infty}$ is eventually in U.

By definition it is clear that every semi-stratifiable space is semi-stratifiable(mod K).

Theorem 3.5. *Every closed subspace of a semi-stratifiable(mod K) space is semi-stratifiable(mod K).*

Proof. Let $X = (X, \mathcal{K}, \mathcal{g})$ be a semi-stratifiable (mod K) space. For a closed subspace Y of X, set $\mathcal{K}' = \{K \cap Y \mid K \in \mathcal{K}\}$ and $\mathcal{g}' = \{G \cap Y \mid G \in \mathcal{g}\}$. Then $Y = (Y, \mathcal{K}', \mathcal{g}')$ is semi-stratifiable(mod K).

The following Theorem 3.6 is another representation of Theorem 4.5 [3].

Theorem 3.6. *A topological space is semi-developable(mod K) if and only if it is first-countable(mod K) and semi-stratifiable (mod K).*

Theorem 3.7. *Every semi-stratifiable(mod K) T_1-space is a β-space.*

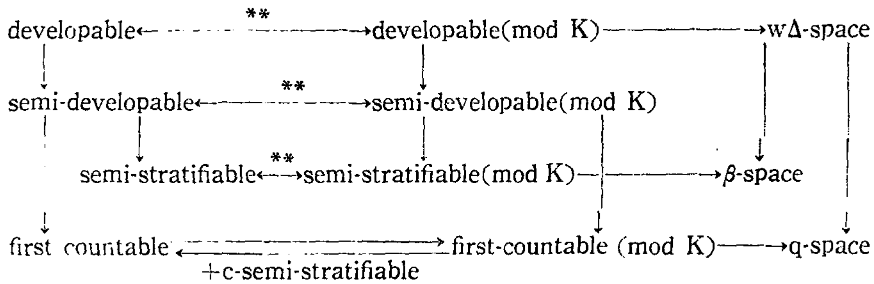
Proof. Suppose $\mathcal{g}$ is an open covering of X and g is a COC-map satisfying the condition in Definition 3.4. We will show that; if x is a point in X and $\{x_n\}_{n=1}^{\infty}$ is a sequence of points in X with $x \in g(n, x_n)$ for all $n \in N$, then $\{x_n\}_{n=1}^{\infty}$ has a cluster point.

Suppose $x \in g(n, x_n)$ for all $n \in N$ but $\{x_n\}_{n=1}^{\infty}$ has no cluster point. Then $\{x_n\}_{n=1}^{\infty}$ is closed set. Consider a compact set K with $x \in K \in \mathcal{K}$. Since $K \cap \{x_n\}_{n=1}^{\infty}$ is finite, we may assume $K \cap \{x_n\}_{n=1}^{\infty} = \phi$. Hence $x \in K \subset X - \{x_n\}_{n=1}^{\infty}$. Since $X - \{x_n\}_{n=1}^{\infty}$ is open, $\{x_n\}_{n=1}^{\infty}$ is eventually in $X - \{x_n\}_{n=1}^{\infty}$. This is a contradiction.

Corollary *A regular T_1-space is semi-stratifiable if and only if it is c-semistratifiable and semi-stratifiable(mod K).*

Proof. By Martin [6], a regular space is semi-stratifiable if and only if it is a c-semi-stratifiable β-space.

Combining the results in [2], [3] and this note, we obtain the following diagram.



All spaces are regular and T_1.

'A $\overset{**}{\longleftrightarrow}$ B' means that A is equivalent to B and c-semi-stratifiable.

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