Group Ring Satisfying A Polynomial Identity

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Let K $[\zeta_1, \zeta_2 \cdots]$ be the polynomial ring over a field K in the noncommuting indeterminates $\zeta_1, \zeta_2, \cdots$. An algebra E over K is said to satisfy a *polynomial identity,* if there exists $f(\zeta_1, \zeta_2, \cdots, \zeta_n) \in K[\zeta_1, \zeta_2, \cdots]$, $f \neq 0$, with

$$f(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$$

for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in E$.

The *standard polynomial of degree n* is defined by

$$s_n(\zeta_1, \zeta_2, \cdots, \zeta_n) = [\zeta_1, \zeta_2, \cdots, \zeta_n] = \sum_{\sigma \in S_n} (-1)^\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

Here s_n is the symmetric group of degree n and $(-1)^\sigma$ is 1 or -1 according as σ is an even or odd permutation.

Kaplansky[2], and Amitsur[3] proved that the group ring K[G] over field K satisfies a nontrivial polynomial identity, if the group G has an abelian subgroup with finite index in G.

In this paper we will find a necessary and sufficient condition for K[G] to satisfy a polynomial identity.

The following three lemmas were proved in [1].

Lemma 1. *Suppose E is an algebra over a field K which satisfies a nontrivial polynomial identity of degree n. Then E satisfies the polynomial identity* $f \in K[\zeta_1, \zeta_2, \cdots, \zeta_n]$ *with*

$$f = \sum_{\sigma \in S_n} a_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

where $a_\sigma \in K$ *and they are not all zero.*

Lemma 2. *Let* $E = K_m$ *be the ring of* $m \times m$ *matrices over K. Then E does not satisfy a polynomial identity of degree less than 2m.*

Lemma 3. K_m*, the ring of* $m \times m$ *matrices over K, satisfies the standard polynomial identity of degree 2m.*

We begin our study of group rings which satisfy a polynomial identity. The following lemma gives a sufficient condition for this to occur.

Lemma 4. *Let group G have an abelian subgroup A with $[G:A]=n<\infty$. Then $K[G]$ satisfies the standard polynomial identity of degree $2n$.*

Proof. Let $x_1, x_2, \cdots, x_n$ be a set of right coset representatives for A in G. Let $E=K[A]$ and $V=K[G]$. Then clearly V is a left E-module with basis $\{x_1, x_2, \cdots, x_n\}$. Now V is also a faithful right K[G]-module. Since right and left multiplication commute as operators on V, it follows that K[G] is a set of E-linear transformations on an n-dimensional free E-module V. This $K[G]\subseteq E_n=E\otimes_k K_n$. By Lemma 3, K_n satisfies s_{2n}. Furthermore s_{2n} is multilinear and E is a commutative ring, so clearly $E_n=E\otimes_k K_n$ also satisfies s_{2n}. Since $K[G]\subseteq E_n$ the result follows.

Now we will investigate some properties of groups and group rings.

Lemma 5. *Let $H_1, H_2, \cdots, H_n$ be subgroups of a group G of finite index. Then $H=H_1\cap H_2\cap\cdots\cap H_n$ has finite index in G and in fact*

$$[G:H]\leq[G:H_1][G:H_2]\cdots[G:H_n]$$

Proof. If Hx is a coset of H, then clearly $Hx=H_1x\cap H_2x\cap\cdots\cap H_nx$. Since there are at most $[G:H_1][G:H_2]\cdots[G:H_n]$ choices for $H_1x, H_2x, \cdots, H_nx$, the result follows.

Lemma 6. *Let G be a group and let $H_1, H_2, \cdots, H_n$ be a finite number of subgroups. Suppose that there exists a finite collection of elements $x_{ij}\in G$ $(i=1, 2, \cdots, n,\ j=1, 2, \cdots, f(i))$ with*

$$G=\bigcup_{i,j} H_i x_{ij}$$

a set theoretic union. Then for some i, $[G:H_i]<\infty$.

Proof. By relabeling we can assume all the H_i to be distinct. We prove the result by induction on n, the number of distinct H_i. The case n=1 is clear.

If a full set of cosets of H_n appears among the H_nx_{nj}, then $[G:H_n]<\infty$ and we are finished. Otherwise if H_nx is missing, then $H_nx\subseteq\bigcup_{i,j} H_ix_{ij}$, But $H_nx\cap H_nx_{nj}$ is empty so $H_nx\subseteq\bigcup_{i\neq n,j} H_ix_{ij}$. Thus

$$H_nx_{nj}\subseteq\bigcup_{i\neq n,j} H_ix_{ij}x^{-1}x_{nj}$$

and G can be written as a finite union of cosets of $H_1, H_2, \cdots, H_{n-1}$ By induction $[G:H_i]<\infty$ for some $i=1, 2, \cdots, n-1$ and the result follow.

Lemma 7. *Let G be a group and suppose that G can be written as $G=\bigcup H_ix_{ij}$, a finite union of cosets.. Then $G=\bigcup' H_ix_{ij}$ where the union is restricted to those H_i with $[G:H_i]<\infty$.*

Proof. Let $S=\{i\mid[G:H_i]<\infty\}$ and $T=\{i\mid[G:H_i]=\infty\}$. By Lemma 6, $S\neq\phi$. Let $W=\bigcap_{i\in S} H_i$. Then $[G:W]<\infty$ by lemma 5 and each coset H_ix_{ij} with $i\in S$ is a finite union of cosets of W. Thus

$$\bigcup' H_ix_{ij}=\bigcup_{i\in S}H_ix_{ij}=\bigcup Wy,$$

a finite union of cosets of W. If $G \neq \cup' H_i x_{ij}$ then $G \neq \cup W_x$, and some coset W_y is missing. Then

$$W_y \subseteq (\cup W_{y_i}) \cup (\cup_{i \in T} H_i x_{ij})$$

and since $W_y \cup W_{y_i}$ is empty we have $W_y \subseteq \bigcup_{i \in T} H_i x_{ij}$.

Thus all cosets of W are contained in finite unions of cosets of H_i with $i \in T$. Since $[G:W] < \infty$, this yields a representation of G as a finite union of cosets of those H_i with $i \in T$. This contradicts Lemma 6, and thus $G = \cup' H_i x_{ij}$.

Lemma 8. *Let G be a group with a central subgroup Z of finite index. Then G′, the commutator subgroup of G, is finite.*

Proof. Let $(x, y) = x^{-1}y^{-1}xy$ denote commutators in G. Since $(x, y)^{-1} = (y, x)$, we see that G′ is the set of all finite products of commutators and it is unnecessary to consider inverses. Let $x_1, x_2, \cdots, x_n$ be coset representatives for Z in G and set $c_{ij} = (x_i, x_j)$. We observe first that these are all the commutators of G. Let x, $y \in G$ and say $x \in Z_{x_i}$, $y \in Z_{x_j}$. Then $x = ux_i$, $y = vx_j$ with u and v central in G. This yields easily $(x, y) = (x_i, y_j) = c_{ij}$.

Now let $x, y \in G$. Since Z is normal in G and G/Z has order n, we have $(x, y)^n \in Z$. Thus

$$\begin{aligned}(x, y)^{n+1} &= x^{-1}y^{-1}xy(x, y)^n \\ &= x^{-1}y^{-1}x(x, y)^n y \\ &= x^{-1}y^{-1}x(x^{-1}y^{-1}xy)(x, y)^{n-1}y \\ &= x^{-1}y^{-2}xy^2 \cdot y^{-1}(x, y)^{n-1}y \\ &= (x, y^2)(y^{-1}xy, y)^{n-1}\end{aligned}$$

since conjugation by y being an automorphism of G implies that

$$\begin{aligned}y^{-1}(x, y)^{n-1}\ y &= (y^{-1}xy,\ y^{-1}yy)^{n-1} \\ &= (y^{-1}xy, y)^{n-1}.\end{aligned}$$

We show finally that every element of G′ can be write as a product of at most n^3 commutators and this will yield the result. Suppose $u \in G'$ and $u = c_1c_2\cdots c_m$, a product of m commutators. If $m > n^3$, then since there at most n^2 distinct c_{ij}, it follows that some c_{ij}, say $c = (x, y)$, occurs at least n+1 times. We shift n+1 of these successively to the left using

$$\begin{aligned}(x_r, x_s)(x, y) &= (x, y)c^{-1}\ (x_r, x_s)c \\ &= (x, y)(c^{-1}x_r c, c^{-1}x_s c)\end{aligned}$$

and obtain $u = (x, y)^{n+1}c'_{n+2}c'_{n+3}\cdots c'_m$ where c'_i is a possibly new commutator. Using

$$(x, y)^{n+1} = (x, y^2)(y^{-1}xy, y)^{n-1}$$

we can write u as a product of m−1 commutators. Thus every element of G′ is a product of at most n^3 of c_{ij} and thus clearly G′ is finite.

Let G be a group. We define

$$\Delta = \Delta(G) = \{x \mid x \in G,\ [G:C_G(x)] < \infty\}.$$

Since the conjugates of x are in one-to-one correspondence with the right cosets of $C_G(x)$, it follows that x has only finitely many conjugates if and only if $x \in \Delta$, and Δ is a normal subgroup of G.

Let θ denote the projection $\theta: K[G] \to K[\Delta]$ given by

$$\alpha = \sum_{x \in G} k_x x \to \theta(\alpha) = \sum_{x \in \Delta} k_x x$$

Then θ is clearly a K-linear map but it is certainly not a ring homomorphism in general.

Lemma 9. *Let H be a finitely generated subgroup of $\Delta(G)$. Then $[H:Z(H)]$ and $|H'|$ are finite. Thus if $\Delta(G)$ contains no nonidentity elements of finite order, then $\Delta(G)$ is torsion free abelian.*

Proof. Let H be generated by $x_1, x_2, \cdots, x_n$. Since each x_i has only a finite number of conjugates in G, they have only a finite number of conjugates in H. Hence $[H:C_H(x_i)] < \infty$. By Lemma 5, $Z = \cap C_H(x_i)$ has finite index in H. Since $x_1, x_2, \cdots, x_n$ generates H, we see that Z is central in H. Thus $[H:Z(H)]$ is finite and by Lemma 8, H' is finite. Now suppose $\Delta(G)$ has no nontrival elements of finite order and let $x, y \in \Delta(G)$. Set $H = \langle x, y \rangle$. Since H is finitely generated subgroup of $\Delta(G)$, the above implies that H' is finite and hence $H' = \langle 1 \rangle$. Thus x and y commute and $\Delta(G)$ is abelian. By definition $\Delta(G)$ is torsion free.

Lemma 10. *Let $G \neq \cup H_m g_{mn}$, a finite union of cosets. Let $\alpha_1, \alpha_2, \cdots, \alpha_s, \beta_1, \beta_2, \cdots, \beta_s \in K[G]$ and suppose that for all $x \in G - \cup H_m g_{mn}$ we have*

$$\alpha_1 x \beta_1 + \alpha_2 x \beta_2 + \cdots + \alpha_s x \beta_s = 0$$

Then there exists $y \in G$ with

$$\theta(\alpha_1)^y \beta_1 + \theta(\alpha_2)^y \beta_2 + \cdots + \theta(\alpha_s)^y \beta_s = 0$$

Poof. Let W be the intersection of the centralizers of all the elements in Supp $\theta(\alpha_i)$ for all $i = 1, 2, \cdots, s$. By Lemma 5, $[G:W] = t < \infty$. Clearly if $x \in W$, then x centralizes $\theta(\alpha_1), \theta(\alpha_2), \cdots, \theta(\alpha_s)$. Let $\{u_i\}$ be a set of right coset representatives for W in G. Let us suppose by way of contradiction that for $j = 1, 2, \cdots t$,

$$\gamma_i = \theta(\alpha_1)^{u_i} \beta_1 + \theta(\alpha_2)^{u_i} \beta_2 + \cdots + \theta(\alpha_s)^{u_i} \beta_s \neq 0$$

and let $v_j \in \text{Supp } \gamma_i$

Write $\alpha_j = \theta(\alpha_j) + \alpha_j'$ where $\text{Supp } \alpha_j' \cap \Delta = \phi$ and then write the finite sums

$$\alpha_j' = \sum a_{jk} y_k \quad y_k \notin \Delta$$

$$\beta_j = \sum b_{jk} z_k$$

with $a_{jk}, b_{jk} \in K$ and y_k, $z_k \in G$. If y_i is conjugate to come $v_i z_k^{-1}$ in G, choose $h_{ijk} \in G$ with $h_{ijk}^{-1} y_j h_{ijk} = v_i z_k^{-1}$

Let $x \in G$ and suppose that $x \notin H_m g_{mn}$. Then we must have

$$\begin{aligned} 0 &= x^{-1}\alpha_1 x \beta_1 + x^{-1}\alpha_2 x \beta_2 + \cdots + x^{-1}\alpha_s x \beta_s \\ &= [\theta(\alpha_1)^x \beta_1 + \theta(\alpha_2)^x \beta_2 + \cdots + \theta(\alpha_s)^x \beta_s] + [\alpha'^x_1 \beta_1 + \alpha_2^{-x} \beta_2 + \cdots + \alpha_s'^x \beta_s] \end{aligned}$$

Since $\{u_i\}$ is a full set of coset representives for W in G, we have $x \in Wu_i$ for some i. Since W centralizes $\theta(\alpha_1), \theta(\alpha_2), \cdots, \theta(\alpha_s)$, the first expression above is equal to γ_i. Hence

$$0 = \gamma_i + [\alpha_1'^x \beta_1 + \alpha_2'^x \beta_2 + \cdots + \alpha_s'^x \beta_s]$$

Now v_i occurs in the support of γ_i and so this element must be canceled by something from the second term. Thus there exists y_j, z_k with $v_i=y_j^x z_k$ or $x^{-1}y_jx=v_iz_k^{-1}=h_{ijk}^{-1}y_jh_{ijk}$. Thus $x\in C_G(y_j)h_{ijk}$. We have therefore shown that

$$G=(\cup H_m g_{mn})\cup(\cup C_G(y_j)h_{ijk}),$$

a finite union of cosets. Now $y_j\notin\Delta$ so $[G:G_G(y_j)]=\infty$ Since by Lemma 7 we can delete subgroups of infinite index from the above union, we have $G=\cup H_m g_{mn}$, a contradiction. The lemma is proved.

Let $K[\zeta_1,\zeta_2,\cdots]$ be the polynomial ring over K in the noncommuting indeterminates $\zeta_1,\zeta_2,\cdots$. A linear monomial is an element $\mu\in K[\zeta_1,\ \zeta_2,\cdots]$ of the form $\mu=\zeta_{i_1}\zeta_{i_2}\cdots\zeta_{i_r}$ with all i_j distinct and with $r\geq 1$. Thus μ is linear in each variable.

Lemma 11. *The number of linear monomials in $K[\zeta_1,\zeta_2,\cdots,\zeta_m]$ is less than or equal to $(m+1)!$.*

Proof. The number of linear monomials in $K[\zeta_1,\zeta_2\cdots,\zeta_m]$ of degree m is of course $m!$. Now any other linear monomial is clearly just an initial segment of one of these. This yields the bound $m\cdot m!\leq(m+1)!$.

Lemma. 12. *Let $K[G]$ satisfy a nontrivial polynomial identity of degree n. Then $[G:\Delta]\leq n!$.*

Proof. We assume by way of contradiction that $[G:\Delta]>n!$. By Lemma 1 we may assume that $K[G]$ satisfies the polynomial identity.

$$f(\zeta_1,\zeta_2,\cdots,\zeta_n)=\zeta_1\zeta_2\cdots\zeta_n+\sum_{\substack{\sigma\in S_n\\ \sigma\neq 1}}a_\sigma\zeta_{\sigma(1)}\zeta_{\sigma(2)}\cdots\zeta_{\sigma(n)}$$

so that clearly $n>1$. For $j=1,2,\cdots,n$ define $f_j\in K[\zeta_j,\zeta_{j+1},\cdots\zeta_n]$ by

$$f=\zeta_1\zeta_2\cdots\zeta_{j-1}f_j+\text{terms not starting with }\zeta_1\zeta_2\cdots\zeta_{j+1}.$$

Then clearly $f_1=f$, $f_n=\zeta_n$, and f_j is a homogeneous multilinear polynomial of degree $n-j+1$. In particular, for all j, ζ_j occurs in each monomial of f_j. We clearly have

$$f_j=\zeta_jf_{j+1}+\text{terms not starting with }\zeta_j.$$

For $j=2,3,\cdots n$, let M_j denote the set of all linear monomials in $K[\zeta_j,\zeta_{j+1},\cdots,\zeta_n]$ and let M_1 be empty. Then by Lemma 11 we have for all j. $|M_j|\leq|M_2|\leq n!$. We show now by induction on $j=1,2,\cdots,n$ that for any $x_j,x_{j+1},\cdots,x_n\in G$, then either $f_j(x_j,x_{j+1},\cdots,x_n)=0$ or $\mu(x_j,x_{j+1},\cdots x_n)\in\Delta$ for some $\mu\in M_j$. Since $f=f_1$ is a polynomial identity satisfied by $K[G]$, the result for $j=1$ is clear.

Suppose the result holds for some $j<n$. Fix $x_{j+1},x_{j+2},\cdots,x_n\in G$ and let $x\in G$ play the role of the j^{th} variable. Let $u\in M_{j+1}$. If $\mu(x_{j+1},x_{j+2},\cdots,x_n)\in\Delta$, we are done. Thus we may assume that $\mu(x_{j+1},x_{j+2},\cdots,x_n)\notin\Delta$, for all $\mu\notin M_{j+1}$. Set $M_j-M_{j+1}=T_j$. Now let $\mu\in T_j$ so that μ involves the variable ζ_j. Write $\mu=\mu'\zeta_j\mu''$ where μ' and μ'' are monomials in $K[\zeta_{j+1},\zeta_{j+2},\cdots,\zeta_n]$. Then $\mu(x,x_{j+1},x_{j+2},\cdots,x_n)\in\Delta$ if and only if

$$x\in\mu'(x_{j+1},\cdots,x_n)^{-1}\Delta\mu''(x_{j+1},\cdots,x_n)^{-1}=\Delta h_\mu$$

a fixed coset of Δ, since μ' and μ'' do not involve ζ_j and since Δ is normal in G. Thus it follows that for all $x \in G - \bigcup_{\mu \in T_j} \Delta h_\mu$, we have $\mu(x, x_{j+1}, \cdots, x_n) \notin \Delta$ for all $\mu \in M_j$, since $M_j \subseteq M_{j+1} \cup T_j$. Since the inductive result holds for j, we conclude that for all $x \in G - \bigcup_{\mu \in T_j} \Delta h_\mu$ we have $f_j(x, x_{j+1}, \cdots, x_n) = 0$. Note that $|T_j| \leq |M_j| \leq n!$ and $[G : \Delta] > n!$ by assumption, so $G - \bigcup_{\mu \in T_j} \Delta h_\mu$ is nonempty. Write

$$f_j(\zeta_j, \zeta_{j+1}, \cdots, \zeta_n) = \zeta_r f_{j+1} + \sum_r \eta_r \zeta_j \eta_r'$$

where $\eta_r, \eta_r' \in K[\zeta_{j+1}, \zeta_{j+2}, \cdots \zeta_n]$ and η_r is a linear monomial. Hence $\eta_r \in M_{j+1}$. Now by the above we have

$$0 = 1 \cdot x \cdot f_{j+1}(x_{j+1}, \cdots, x_n) + \sum_r \eta_r(x_{j+1}, \cdots, x_n) \cdot x \cdot \eta_r'(x_{j+1}, \cdots, x_n)$$

for all $x \in G - \bigcup_{\mu \in T_j} \Delta h_\mu \neq \phi$. Hence by Lemma 10 there exists $y \in G$ with

$$0 = \theta(1) y_{j+1}(x_{j+1}, \cdots, x_n) + \sum_r \theta(\eta_r(x_{j+1}, \cdots, x_n)^y \eta_r'(x_{j+1}, \cdots, x_n),$$

Clearly $\theta(1)^y = 1$. Also $\eta_r(x_{j+1}, \cdots, x_n) \in G - \Delta$ since $\eta_r \in M_{j+1}$ and hence $\theta(\eta_r(x_{j+1}, \cdots, x_n)) = 0$. Thus

$$0 = 1 \cdot f_{j+1}(x_{j+1}, \cdots, x_n) = f_{j+1}(x_{j+1}, \cdots, x_n)$$

and the induction step is proved. In particular. the inductive result holds for $j = n$. Here $f_n(\zeta_n) = \zeta_n$ and $M_n = \{\zeta_n\}$. Thus we conclude that for all $x \in G$ either $x = 0$ or $x \in \Delta$, a contradiction since $G \neq \Delta$. Therefore the assumption that $[G : \Delta] > n!$ is false and the lemma is proved.

Theorem. *Let G be a group, and let Δ be finitely generated. Then the group ring K[G] over field K satisfies a polynomial identity over K if and only if* $[G : \Delta] < \infty$.

Proof. If K[G] satisfies a polynomial identity, then $[G : \Delta] < \infty$ by Lemma 12.

Conversely, suppose that $[G : \Delta] < \infty$. Then by Lemma 9, $[\Delta : Z(\Delta)] < \infty$ and $[G : Z(\Delta)] = [G : \Delta][\Delta : Z(\Delta)] < \infty$. Hence $Z(\Delta)$ is an abelian group of finite index in G. Therefore K[G] satisfies a polynomial identity by Lemma 4.

REFERENCES

[1] Amitsur, S. A., and Levitzki, J., (1950), Minimal identities for algebras. *Proc. Amer. Math. Soc.* **1**, 449-463

[2] Kaplansky, I., (1949), Groups with representations of bounded degree. *Canad. J. Math.*, **1**, 105-112

[3] Amitsur, S. A., (1961), Groups with representations of bounded degree II. *Illinois J. Math.* **5**, 198-205

[4] Passman, D. S., (1971), *Infinite group rings.* Marcel Dekker Inc.