Partially Balanced Bipartite Weighing Designs from Association Schemes

Kishore Sinha*

1. Introduction

Bose and Cameron [1], [2] developed some methods of constructing balanced weighing designs (BWD). Surayanarayana [6], [7] constructed partially balanced weighing designs (PBWD's) with two and three associate classes, which arose as extensions of the balanced weighing designs. Sinha and Saha [5] constructed some series of PBWD's with two and m-associate association schemes of triangular type. Huang [3] introduced balanced bipartite weighing designs as extensions of BWD and it will be further extended to define partially balanced bipartite weighing designs. A method of construction of partially balanced bipartite weighing designs (PBBWD) from association schemes is given. Also, some series of three associate PBWD's are obtained from three associate rectangular association scheme [4].

2. Definition: Partially Balanced Bipartite Weighing Design

We shall recall the definition of m-associate association scheme from [4].

Definition 1.1. Given v objects a relationship satisfying the following conditions is said to be an association scheme with m-classes:

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1. Any two objects are either 1st, 2nd, ..., or m-th associates, the relation of association being symmetrical; that is, if the symbol α is the i-th associate of the symbol β, then β is the i-th associate of α.
2. Each object α has n_i i-th associates, the number being independent of α.
3. If any two objects α and β are i-th associates, then the number of objects that are j-th associates of α and k-th associates of β, is p^i_{jk} and is independent of the pair of i-th associates α and β.

The numbers v, $n_i(i=1, 2, \ldots, m)$ and p^i_{jk} are called the parameters of the association scheme.

Given an association scheme for the v objects, we define a partially balanced bipartite weighing design (PBBWD) as follows:

Definition 1. 2. If we have an association scheme with m classes if the v objects are arranged into b blocks of size $k(=k_1+k_2:<v)$ such that

1. Every object occurs at most once in a block.
2. Every object occurs in exactly r blocks and every block denotes a set of $k(=k_1+k_2)$ objects which are divided into two subsets of sizes k_1 and k_2.
3. If two objects α and β are i-th associates, then they occur together in the same subsets λ_{1i} times and in opposite subsets $\lambda_{2i}(i=1, 2, \ldots, m)$ times.

The numbers $v, b, r, k_1, k_2, \lambda_{1i}, \lambda_{2i}(i=1, 2, \ldots, m)$ are called the parameters of the PBBWD.

A partially balanced bipartite weighing design (PBBWD) with $k_1=k_2=p$ is a partially balanced weighing design (PBWD).

3. The Method:

Let A be an m-class association scheme with the parameters $(v, n_i, p^i_{jk}\ ; i, j, k = 1, 2, \ldots, m)$ and let Bi $(i=0, 1, \ldots, m; Bo=I)$ be the i-th association matrix of A.

When the i-th and j-th $(i, j=1, 2, \ldots, m; i\neq j)$ associates of a treatment with or without the treatments itself (in 1st subset) constitutes the 1st and 2nd subsets of a block, we get PBBWD's with the original association scheme. Thus we can establish:

Theorem 1. If there exists an m-class association scheme, then there exists PBBWD's with the parameters

(1.a) $v=b,\ \ r=n_i+n_j+1=k,\ \ k_1=n_i+1,\ \ k_2=n_j,$

$$\lambda_{1u}=p^u_{ii}+p^u_{jj},\ \ \lambda_{2u}=2p^u_{ij},\ \ u=1,2,\ldots,m;\ \ u\neq i,j,$$

$$\lambda_{1i}=p^i_{ii}+p^i_{jj}+2,\ \ \lambda_{2i}=2p^i_{ij},$$

$$\lambda_{1j}=p^j_{ii}+p^j_{jj},\ \ \lambda_{2j}=2p^j_{ij}+2.$$

(1.b) $v=b,\ \ r=n_i+n_j=k,\ \ k_1=n_i,\ \ k_2=n_j,$

$$\lambda_{1u}=p^u_{ii}+p^u_{jj},\ \ \lambda_{2u}=2p^u_{ij},\ \ u=1,2,\ldots,m.$$

Proof: (a) We construct a design given by the incidence matrix

$N=[I+B_i+B_j]$ where I is the $(v\times b)$ identity matrix,

$$NN'=[I+B_i+B_j][I+B_i'+B_j']$$

$$=(n_i+n_j+1)B_0+\sum_{\substack{u=1\\u\neq i,j\\i\neq j}}^{m}(p^u_{ii}+p^u_{jj}+2p^u_{ij})B_u$$

$$+(p^i_{ii}+p^i_{jj}+2p^i_{ij}+2)B_i+(p^j_{ii}+2p^j_{ij}+p^j_{jj}+2)B_j.$$

Therefore, the PBIB design whose incidence matrix is N has

$$\lambda_u=p^u_{ii}+p^u_{ij}+2p^u_{ij},\ \ u=1,2,\ldots,m;\ \ u\neq i,j$$

$$\lambda_w=p^w_{ii}+p^w_{jj}+2p^w_{ij}+2;\ w=i,j.$$

Hence it follows (*cf.* [7]) easily that the PBWD has

$$\lambda_{1u}=p^u_{ii}+p^u_{jj},\ \ \lambda_{2u}=2p^u_{ij},\ \ u=1,2,\ldots,m;\ \ u\neq i,j$$

$$\lambda_{1i}=p^i_{ii}+p^i_{jj}+2,\ \ \lambda_{2i}=2p^i_{ij}$$

$$\lambda_{1j}=p^j_{ii}+p^j_{jj},\ \ \lambda_{2j}=2p^i_{ij}+2.$$

(b) We construct a design given by the incidence matrix

$$N=[B_i+B_j].$$

Here $NN'=[B_i+B_j][B_i'+B_j']=\sum_{u=0}^{m}(p_{ii}^u+2p_{ij}^u+p_{jj}^u)B_u$

$$=(n_i+n_j)B_0+\sum_{u=1}^{m}(p_{ii}^n+2p_{ij}^u+p_{jj}^u)B_u.$$

Thus the PBIB design with the incidence matrix N has

$$\lambda_u=p_{ii}^u+2p_{ij}^u+p_{jj}^u.$$

Hence it follows (*cf.* [7]) that the PBWD has $\lambda_{1u}=p_{ii}^u+p_{jj}^u$ and $\lambda_{2u}=2p_{ij}^u$. Hence the theorem is proved.

Corollary. There exists the following series of PBWD designs with the rectangular association scheme and with the parameters

(1. i. b) $v=n^2=b,\ m=n,\ r=2(n-1)=k,\ k_1=k_2=n-1,$
$\lambda_{11}=n-2,\ \lambda_{12}=n-2,\ \lambda_{13}=0,$
$\lambda_{21}=0,\ \lambda_{22}=0,\ \lambda_{23}=2;$ for $i=1,\ j=2,$

(1. ii. b) $v=2n=b,\ m=2,n,\ r=2(n-1)=k,\ k_1=k_2=n-1,$
$\lambda_{11}=n-2,\ \lambda_{12}=0,\ \lambda_{13}=0$
$\lambda_{21}=0,\ \lambda_{22}=0,\ \lambda_{23}=2;$ for $i=2,\ j=3,$

(1. i. a) $v=n(n+1)=b,\ m=n+1,n,\ r=2(n)=k,\ k_1=k_2=n,$
$\lambda_{11}=n,\ \lambda_{12}=n-1,\ \lambda_{13}=0,$
$\lambda_{21}=0,\ \lambda_{22}=2,\ \lambda_{23}=2;$ for $i=1, j=2,$

by forming blocks with 1st subset from i-th associates with (in 1. i. a) or without (in 1. i-ii. b) the treatment itself and 2nd subset from j-th associates.

ABSTRACT

A method of constructing partially balanced bipartite weighing designs (PBBWD) from association schemes is given. Some series of three associate PBBWD's from three associate rectangular association schemes are obtained.

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