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Optimal Control of Electrohydraulic Actuator System

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電氣油壓式 액튜에이터의 最適制御

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要 約

電氣油壓式 액튜에이터의 時間領域內에서의 性能을 向上시키기 爲해서 最適制御理論을 適用하였다. 系統의 定量的인 性能을 二次性能指數로 表示하였다.

油壓 액튜에이터와 밸브의 線形的인 傳達特性을 利用해서 系統의 狀態方程式을 세우고 리카티 方程式의 解를 컴퓨터로 求해서 系統의 最適入力을 決定하였다.

以上과 같이 만들어진 最適制御系統의 變位, 速度, 加速度의 過度應答을 求하기 爲해서 아날로그 컴퓨터를 使用하고 그 應答과 P.W.M. 系統의 應答을 比較한 結果, 最適制御系統이 더욱 빠르고 安定된 應答을 나타냄을 알았다.

그 比較를 具體的이고 定量的으로 行하기 爲해서 性能指數曲線을 求해서 比較한 結果 그 性能指數의 척도로 볼때 最適制御系統이 P.W.M.系統보다 約 35% 까지 優秀하다는 結論을 얻었다.

1. INTRODUCTION

Electrohydraulic servomechanisms are commonly used where the requirements exist for a sufficient accuracy and quick response with high power. Several approaches have been conducted in this area to improve the performances of the servomechanisms such as the acceleration switching servomechanisms and the pulse width modulated servomechanisms[1,2].

These servos employ essentially the method that appropriately modulates the error signal and transfer in the form of electrical input (voltage, current) to the valve.

In conventional electrohydraulic servomechanisms, the outputs are fed back by potentiometer with an appropriate feedback gain. These are compared with the input values of the overall closed-loop system and the error signals are, after amplification, sent to the plant again[3].

Then there arises the question that what values of the feedback gain should be used to get the best overall system response. There is no simple rule to aid the designer to find the feedback gain. The final choice usually results in a compromise of the desirability of rapid responses and the desire to reduce excessive overshoot that garantees, though not directly, the stability of the system. Usually the root-locus technique was used to determine the best

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feedback gains by trial and error. But the technique gives us no information about the optimality. When the system is desired to be controlled with a certain performance index or has more than two inputs, the technique is not applicable. In the above two servo systems, the so-called A.S. servo and P.W.M. servo, the main interest lies in the performance in the frequency domain. For linear systems, it can be shown that frequency response characteristics well agree with transient responses and it is also very important to know the transient responses and to improve them[4].

In this paper, the responses in the time domain are considered and with the optimal control theory performance improvements of the system is investigated. The performance measure such as rapid responses, accuracy and stability are described in the form of quadratic performance index.

To obtain the optimal control policy, Riccati equation is solved and the steady state feedback gains and input gain are computed with this solution. The resulting system responses and performance index curves are computed and compared with those of P.W.M. system with analog computer. In order to obtain numerical values of the system, the characteristics of Moog valve are used.

2. THEORETICAL BACKGROUND

2-1. Optimal control theory

This is the brief of the well known linear tracking problem. The linear plant to be controlled can be described in general as the following vector matrix differential equation;

$$\dot{X}(t) = AX(t) + BU(t) \tag{1}$$

where A is $n\times n$ square matrix, B is $n\times m$ matrix, $X(t)$ is n-dimensional state vector and $U(t)$ is m-dimensional control vector.

In the linear tracking problem, the performance index to be minimized can be selected quadratic form as

$$\begin{aligned} J = & [X(t_f) - r(t_f)]^T \cdot H(t) \cdot [X(t_f) - r(t_f)] \\ & + \int_{t_0}^{t_f} \{[X(t) - r(t)]^T \cdot Q(t) \cdot [X(t) - r(t)] \\ & + U^T(t) \cdot R(t) \cdot U(t)\} dt \end{aligned} \tag{2}$$

where t_0 is initial time and t_f is terminal time and $r(t)$ is the desired value or reference state vector that is not the origin 0. H and Q are real symmetric positive semidefinite matrices, and R is a real symmetric positive definite matrix. The physical interpretation of this index is that it is desired to minimize the deviation of each state from reference state in the time interval $[t_0, t_f]$ and also the deviation of final states without excessive expenditure of control effort.

Assuming that admissible states and control are not bounded, and applying variational method, the optimal law is given as the following;[11].

$$\begin{aligned} U^*(t) = & -R^{-1}(t) B^T(t) K(t) X(t) \\ & -R^{-1}(t) B^T(t) S(t) \end{aligned} \tag{3}$$

where $U^*(t)$ is optimal control input and $K(t)$ and $S(t)$ are the solutions of the following vector matrix differential equations;

$$\begin{aligned} \dot{K}(t) = & -K(t) A(t) - A^T(t) K(t) - Q(t) \\ & + K(t) B(t) R^{-1}(t) B^T(t) K(t) \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{S}(t) = & -[A^T(t) \\ & -K(t) B(t) R^{-1}(t) B^T(t)] S(t) \\ & + Q(t) r(t), \end{aligned} \tag{5}$$

with boundary conditions

$$K(t_f) = H \tag{6}$$

$$S(t_f) = -Hr(t_f) \tag{7}$$

Here equation (4) is called the Riccati equation and these equations can be integrated numerically from t_f to t_0.

By storing $K(t)$ and $S(t)$ in a computer, the optimal input $U^*(t)$ can be determined and the plant is optimally controlled.

2-2. Steady state solution

Kalman has shown if[5]

(a) the plant is completely controllable,

(b) $H=0$

(c) A, B, R and Q are constant matrices, then $K(t) \to K$ (a constant matrix) as $t_f \to \infty$.

From practical viewpoint, it may be feasible to use the fixed control law even for the process of finite time interval. If terminal time t_f is exceedingly large but finite and t_1 and t_2 are also very large such that $0 < t_1 < t_2 < t_f < \infty$, then the matrix $K(t)$ may be approximated by constan matrix K for all $t \epsilon [t_0, t_2]$. In other words

$$K(t) \cong K \text{ for } t \epsilon [t_0, t_2]. \quad (8)$$

Using this approximation, the equation (5) will be

$$S(t) \cong -(A^T - KBR^{-1}B^T)S(t) + Qr(t) \quad (9)$$

If we set

$$G = A - BR^{-1}B^T K \text{ and,}$$
$$r(t) = r, \quad (10)$$

then equation (8) becomes

$$\dot{S}(t) \cong -G^T S(t) + Qr. \quad (11)$$

For approximate system in equation (11), with respect to t_2, the solution will be

$$\dot{S}(t) \cong (G^T)^{-1}Qr + e^{G^T \cdot t2}[(G^T)^{-1}e^{-G^T t}Qr + e^{-G^T t}S(t_2)]$$
$$\text{for all } t \epsilon [t_0, t_2] \quad (12)$$

On the above conditions, all the eigenvalues of the matrix G have negative real part. Thus

$$e^{G^T \cdot t2} \cong 0$$

By approximation, constant matrix for $S(t)$ is given as

$$S = (G^T)^{-1} \cdot Qr \quad (13)$$

By equation (8) and (13), fixed law of optimal control is given as the following.

$$U^* = -R^{-1}B^T K \cdot X(t) - R^{-1}B^T S = F \cdot X(t) + V, \quad (14)$$

where

$$F = -R^{-1}B^T K \text{ and} \quad (15)$$
$$V = -R^{-1}B^T S = -R^{-1}B^T (G^T)^{-1} \cdot Qr = -R^{-1}B^T (A^T - KBR^{-1}B^T)^{-1}Qr \quad (16)$$

2-3. Stability of the optimal system

It is shown that if the linear time invariant system expressed in Eq. (1) with performance index

$$J = \int_{t0}^{tf} [X^T(t)QX(t) + U^T(t)R^{-1}U(t)]dt \quad (17)$$

(a) is controllable and

(b) U is not constrained,

then the matrix G have negative real part eigenvalues[12]. Thus the system variables will have fixed values as the time increases and by the definition of stability, the optimal control system described by equation (1) is stable.

In tracking problem in this paper, as the matrix G is the same in both cases and the stability of linear system is independent of input signal, it can be said that the optimal control system in this case is stable.

3. APPLICATION OF THE THEORY

In order to apply the optimal control law in equation (14), the servovalve and the actuator must be expressed in the form of state equation. This can be done by analysis of the servovalve and the actuator respectively, and state space representation

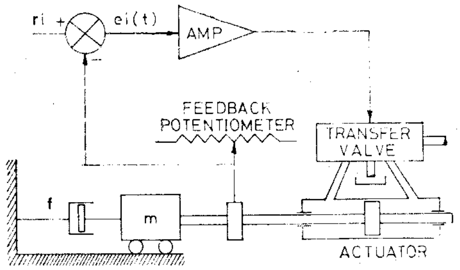

Fig. 1. Schematic diagram of electrohydraulic servo

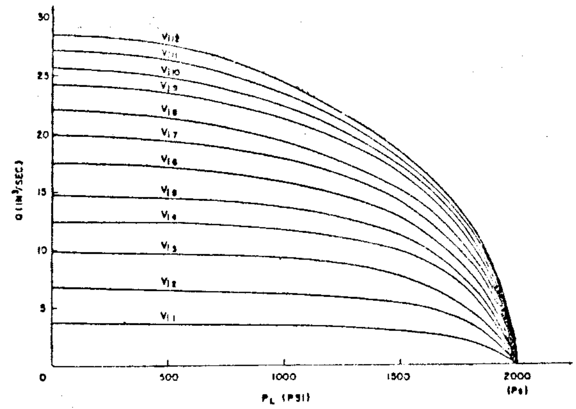

Fig. 3. Typical valve pressure flow characteristics

s made to assume the optimal control policy.

3-1. Analysis of the system

Fig. 1 shows a schematic diagram of typical electrohydraulic servomechanism which is considered in this paper. The transfer valve is a conventional 2-stage 4-way electrohydraulic flow control valve as shown in Fig. 2. The transfer valve employs a hydraulic preamplifier of the flapper type. As can be seen the flapper valve is torque motor actuated and controls two opposed resistances. The load is comprised of mass and viscous friction.

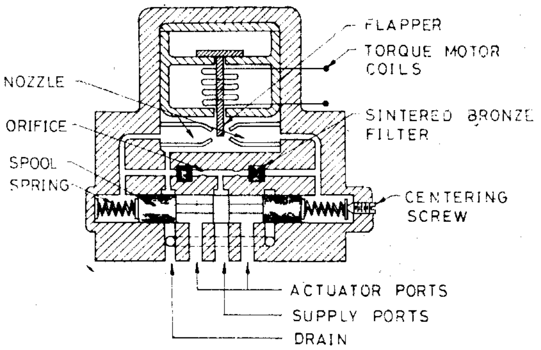

Fig. 2. Electrohydraulic servovalve

With most of parasitic nonlinearities (such as hysteresis, deadzone, stiction etc.) materially reduced, the dynamic characteristics of a 2-stage electrohydraulic servovalve can be represented by a second order transfer function of the form; 6)

$$G_v(s)=\frac{Q_v(s)}{Vi(s)}=\frac{K_v\cdot W_{nv}^2}{S^2+2\xi_{nv}\cdot W_{nv}\cdot s+W_{nv}^2} \quad (18)$$

where q_v the valve flow, Vi valve input voltage, ξ_{nv} valve damping ratio and W_{nv} valve natural frequency and valve flow gain $K_v(q_v/Vi)$ can be determined from the valve pressure-flow characteristics. Fig. 3 shows a typical set of pressure-flow curves. It can be seen from these curves that if the operation of the valve is not large from the origin, the flow gain is almost independent of a control signal and the above linearity is ensured.

As to the actuator, it may be analysed as the following. The load elements of a typical hydraulic actuator consist of inertia, effects of oil compressibility, compliance of fluid lines and load cylinder viscous damping, stiction and coulomb friction. The behavior of some of these load elements, e.g, stiction and Coulomb friction is quite nonlinear. However stiction occurs only when the load starts to move and can be eliminated by applying a suitable dither, while Coulomb friction may be replaced, with reasonable accuracy by an equivalent viscous damping, linear analysis of the load is thus possible. For an inertia and viscous damping load, the equation of motion is given by

$$P_L \cdot A_p = m\frac{d^2x_0}{d\,t^2} + f_v\frac{dx_0}{dt}, \qquad (19)$$

where P_L is load pressure, A_p the piston area, x_0 the load position, m the mass of the load and f_v the friction coefficient.

From the law of conservation of fluid flow, the total valve flow q_v is equal to the sum of fluid flow due to the motion of the piston in the load cylinder q_a, an equivalent flow of compressed flow q_c and leakage flow q_l. Hence,

$$q_v = q_a + q_c + q_l \qquad (20)$$

where

$$q_a = A_p\frac{dx_0}{dt} \qquad (21)$$

$$q_c = \frac{V}{2\beta}\cdot\frac{dP_L}{dt} \qquad (22)$$

$$q_l = K_l \cdot P_L \qquad (23)$$

Here V is one half of total volume contained by actuator and lines between actuator and valve, β oil bulk modulus, and K_l the leakage coefficient.

From equation (19)～(23), q_v becomes

$$q_v = A_p \cdot \dot{x}_0 + \frac{V}{2\beta}\Big[\frac{m}{A_p}\dddot{x}_0 + \frac{f_v}{A_p}\ddot{x}_0 + K_l\frac{m}{A_p}\ddot{x}_0 + \frac{f_v}{A_p}\dot{x}_0\Big]$$

$$= \dot{x}_0\Big(A_p + \frac{f_v}{A_p}K_l\Big) + \ddot{x}\Big(\frac{f_v}{A_p}\cdot\frac{V}{\beta} + \frac{K_l \cdot m}{A_p}\Big) + \dddot{x}_0\frac{mV}{A_p \cdot 2\beta} \qquad (24)$$

If we set

$$A_1 = A_p + (f_v \cdot K_l/A_p)$$
$$A_2 = (f_v/A_p)\cdot(V/2\beta) + (K_l \cdot m/A_p) \qquad (25)$$
$$A_3 = (1/A_p)(mV/2\beta),$$

then equation (24) becomes

$$q_v = A_1\dot{x}_0 + A_2\ddot{x}_0 + A_3\dddot{x}_0 \qquad (26)$$

From the above transfer function and the equation of motion, state variables and input variables are chosen as $x_1 = x_0$, $x_2 = \dot{x}_0$, $x_3 = \ddot{x}_0$, $x_4 = q_v$, $x_5 = \dot{q}_v$ and $U = Vi$. By rearranging equation (26), the state variables are represented as;

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = x_4/A_3 - (A_3/A_3)x_3 - (A_1/A_3)x_2 \qquad (27)$$
$$\dot{x}_4 = x_5$$
$$\dot{x}_5 = Vi \cdot K_v \cdot W_{nv}^2 - W_{nv}^2 \cdot x_4 - 2\xi_{nv} \cdot W_{nv} \cdot x_5$$

Thus the state equation of this system is

$$\dot{X} = AX + BU,$$

where

$$X^T = (x_1\ x_2\ x_3\ x_4\ x_5)$$
$$B^T = (0\ 0\ 0\ 0\ K_v \cdot W_{nv}^2), \text{ and}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -A_1/A_3 & -A_2/A_3 & 1/A_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -W_{nv}^2 & -2\xi_{nv}\cdot W_{nv} \end{pmatrix} \qquad (28)$$

with initial condition

$$X^T(0) = (0\ 0\ 0\ 0\ 0). \qquad (29)$$

Numerical values of system parameters of conventional 2-stage, 4-way servovalve and actuator in this paper are given in Table 1. By substituting these parameters

Table 1. System parameters

$A_p = 0.623\ in^2$	$m = 0.157$ lb-sec^2/in
$\beta = 2.0 \times 10^3$ psi	$V = 2.575\ in^3$
$f_v = 19.86$ lb-sec/in	$\xi_{nv} = 1.05$
$k_l = 1.38 \times 10\ in^5$/lb	$W_{nv} = 214$ rad/sec

Table 2. Dynamic constants for the system

$$A=\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -65.5194 & -22.5108 & 61.64 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4.580 & -4.494 \end{pmatrix}$$

$$B=(0 \quad 0 \quad 0 \quad 0 \quad 13.9678).$$

into equations (25)~(27) and time scaling of system equations in the unit of 1/100 sec, the matrix A and vector B of equation (1) become as the following. (Table 2)

3-2. Application

As controllability is the necessary condition for optimal control in this paper, it is needed first to determine if the system is controllable or not. A linear system is completely controllable if and only if the matrix P where

$$P=[B : AB : A^2B : A^3B : A^4B]$$

is of rank n[7]. The P matrix in this case is calculated to be

$$P=\begin{pmatrix} 0 & 0 & 0 & 0 & 861.54 \\ 0 & 0 & 0 & 861.54 & -5644.81 \\ 0 & 0 & 861.54 & -18533 & 417155.3 \\ 0 & 13.97 & 13.97 & 218.1 & 374151.2 \\ 13.98 & -62.8 & 218.1 & -1044.12 & -5667.2 \end{pmatrix}$$

and the column vectors of P are linearly independent. Thus it is of rank 5, and the system is completely controllable.

In hydraulic servo system, it is considered to be most important to maintain the velocity and the position of the load as close as possible to the desired state in the time interval $[t_0, t_f]$. Hence the performance may be expressed as the following.

$$J=\int_{t_0}^{t_f}[q_{11}\cdot(x_1-r_i)^2+q_{22}(x_2-r_2)^2+r_{11}u^2]dt \qquad (30)$$

where H equal to 0 and r_i, r_2(which equals to 0) are the desired values of the state x_1, x_2 (position and velocity) and q_{11}, q_{22} are elements of weighting matrix Q.

Thus the matrix Q is

$$Q=\begin{pmatrix} q_{11} & 0 & 0 & 0 & 0 \\ 0 & q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (31)$$

By adjusting the element values, we can weigh relative importance of the deviation of each states from their desired values. These values should also be adjusted to normalize the numerical values encountered. We can normalize them by comparing closed-loop transient responses of the states. By examining the results of analog computation of the system in Eq. (1), with an appropriate amplifier gain Ka, we know that the ratio $(x_1-r_i)/(x_2-r_2)$ becomes equal to 2~3 in the time interval $[t_0, t_f]$. Hence to normalize the numerical values, we make the ratio between q_{11} and q_{22} equal to 2~3. The analog simulation is shown in Fig. 4.

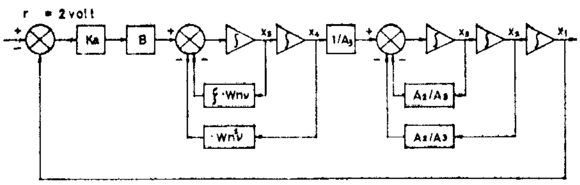

Fig. 4. Analog simulation of the electrohydraulic servo

To avoid placing bounds on admissible controls or to conserve the control energy, we keep r_{11} not zero. In this paper, three

Table 3. F, V obtained

case	f_1	f_2	f_3	f_4	f_5	V
i)	−0.7038	−0.3634	−0.017	−0.7933	−0.1442	0.1408
ii)	−1.580	−1.138	−0.05464	−1.9214	−0.2936	0.3159
iii)	−2.738	−1.875	−0.0894	−2.676	−0.3758	0.5476

cases are considered, i.e. when

i) $q_{11}=1$ $q_{22}=2$

ii) $q_{11}=5$ $q_{22}=15$

iii) $q_{11}=15$ $q_{22}=30$

for each case $r_{11}=2$.

With the system input $r_i=2$ volt, the state equation and the performance index, the Riccati equation (4) and boundary condition (6) are expressed as

$$\dot{K}(t)=K(t)BR^{-1}B^TK(t)-K(t)A-A^TK(t)-Q \qquad (32)$$

$$K(t_f)=0. \qquad (33)$$

Using the method in section 4-1, the steady state solution of the above equation is obtained and by substituting the solution into equation (15) F matrix is determined. By equation (16) and by obtaining the inverse of the matrix G^T, V matrix also can be determined. The solutions are shown in Table 3.

With the system input $r_i=2$ volt (step input) the transient responses are obained with analog computer by the method in section 4-2. They are compared with those of P.W.M. electrohydraulic system. The block diagram of the optimal control system are shown in Fig. 5. Performance index curves are also obtained with analog computation by the method in section 4-2. They are also shown with those of P.W.M. system to compare the theoretical perfor-

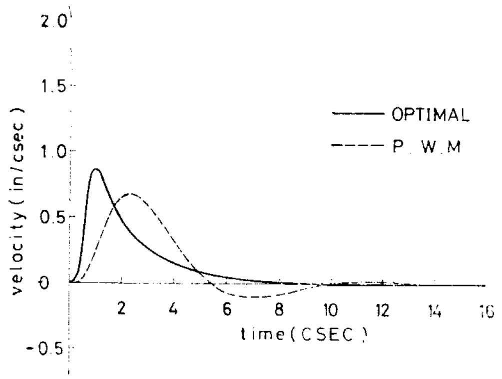

Fig. 6. Transient responses when $q_{11}=2$, $q_{22}=5$

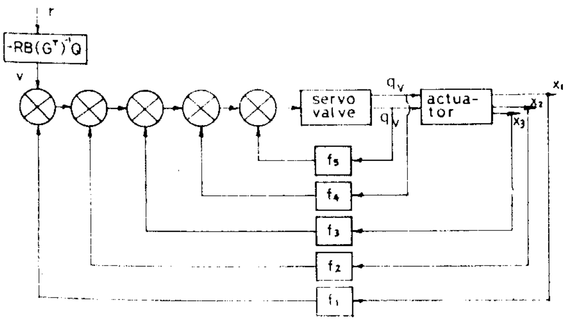

Fig. 5. Block diagram of optimal system

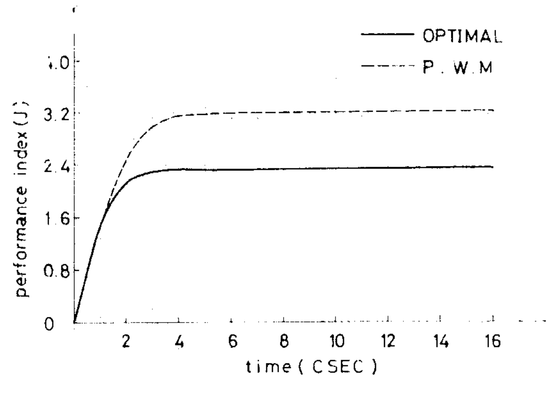

Fig. 7. Transient responses when $q_{11}=5$, $q_{22}=15$

mance concretely.

Among the responses of the five state variables, position, velocity and acceleration of the load with respect to time in cases of i) and ii) among the three cases and the performance index curves are shown in Fig. 6~7.

4. METHODS OF COMPUTATION

4-1. Solution of optimal control law

There are various methods for numerical solution of Riccati equation. Theses are direct integration method, the Kalman-Englar method and the Newton-Raphson method and etc. [8]. In this paper, the Runge-Kutta method is used. By Runge-Kutta method the Riccati equation with final boundary values can be solved backward. It may be regarded as the steady state solution when the difference between the values at a stage and next stage is less than the error bound ε.

after the K matrix and F vector are obtained, $(G^T)^{-1}$ is computed. In computation of K matrix, subroutine "R.K.G.S." which uses Runge-Kutta-Gill method, is applied. The inverse of G^T is obtained with subroutine "M.I.N.V" which employs the standard Gauss-Jordan method.

I.B.M 360 computer is used in this computation[9].

4-2. Transient responses and Performance index

After time scaling is made (1 mechanical sec is equal to 1/100 sec), the plant described in equation (1) is simulated to patch up. To avoid overloadings, magnitude scaling is made. By magnitude scaling, 1 volt in this problem is set 0.25 M.U..

At first, the plant $\dot{X}=AX+BU$ is patched up and with suitable amplitude gain $k_a=0.04534$, and step input $r_i=2$ volt, the closedloop responses are obtained to normalize numerical values of position and velocity. The optimal control system described in the block diagram is patched up by corresponding adding circuits.

To calculate the performance index, equation (30) is also simulated and added to the optimal control system.

By changing F matrix, $\dot{V}$, and q_{11}, q_{22} with potentiometer, the transient responses and performance of the three cases are obtained.

The same procedures are also repeated in the case of P.W.M. system, except that circuits corresponding to the modulator are made instead of optimal controller.

In computation ANDO 401 model is used and an X-Y recorder is employed in recording the computed values.

5. RESULTS AND DISCUSSIONS

The results of this paper are shown in Fig. 6(a)~7(d). Fig. 6(a) shows the transient responses of the displacement $x_0(t)$ of the mass in case i) indicating that the response characteristics of the optimal control system is like the system with dead-beat performance. It has almost no overshoot and oscillation and rapid response having transient period of about 80 msec., while the P.W.M. system has 21% overshoot and oscillation, whose transient period is almost 160 msec. It apparently shows superior response to the P.W.M. system. Fig. 6(b) shows the transient responses of velocity v_0 of the mass in case i), indicating that the deviation of velocity from the reference value 0 is much less than that of the P.W.M. system and goes quickly to 0 without steady state error. The transient period in this case is also 80 msec, while that of P.W.M. is 130 msec. Fig. 6(c) shows the response of the acceleration. The acceleration of the mass is a little larger and sharper than that of the P.W.M. system. This behavior can be expected from the fact that larger acceleration is necessary to decrease the velocity fast. The transient period is the same as those of position and velocity. Fig. 6(d) shows the performance indices of two systems which indicate the results that the value of J described in equation (30) of the optmal system is less than that of P.W.M. system. The reason could be that the performance index J of the equation (30) is used to measure the accumlated deviations and deviations of the state variables of the P.W.M. system from the reference values are larger than those of the optimal system as shown in (a), (b) and (c) of Fig. 6. When q_{11} and q_{22} increase with r_{11} fixed, the deviations of the velocity of the optimal system become smaller and the accleeration curve become sharper accordingly while the responses of the position are almost the same. The transient periods are shortened a little (5 msec).

We may determine from these results that, except for the case when the characteristics of the velocity are needed to be particularly excellent, it is not necessary to increase q_{11} and q_{22}, because larger q_{in} needs more control effort and the tendency of the acceleration to become sharper and more impulsive is not desirable.

From the above results it may be concluded that;

(1) The responses of optimal control system are less oscillatory and rapid than those of the P.W.M. system.

(2) When weighting factors q_{11} and q_{22} increase, the responses of the velocity are a little improved, while the responses of the position remain almost the same.

(3) The increase of the value of the weighting matrix elements is not necessary and desirable.

(4) With the performance index defined in this paper, the optimal control system is by an approximate rate of up to 35% superior to the P.W.M. system.

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