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⟨**original**⟩

The Effects of Longitudinal Inertia Force and Shear Deformation on the Large Amplitude Vibrations of Beams

Nack-Joo Lee*

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軸方向의 慣性力과 剪斷變形이 보의 振動에 미치는 영향

李 樂 周

抄 錄

1端이 軸方向으로 自由로이 움직일 수 있는 單純支持보의 振動에 있어서 그 振幅이 크고, 軸方向의 慣性力과 剪斷變形을 고려할때의 振動을 살펴다. 運動方程式을 처짐을 獨立變數로 하는 非線型偏微分方程式으로 表示하고, modal expansion과 Galerkin方法에 의해서 非線型聯立常微分方程式으로 變型한 다음에 Perturbation method of multiple scales로 近似解를 구하였다. 또한 보의 振動數—振幅의 關係에 대한 一般的 表現을 구하고, 간단한 具體的 例에 대하여 1次近似解와 振動數—振幅關係를 計算하여 이미 이루어진 研究結果와 比較하였다.

1. Introduction

Euler-Bernoulli theory of beam vibration is based upon some restrictive assumptions, such as those for small deflections, where the supports are free to move in the axial direction and the deflection is inextensional, and so on. Effects of shear deformation and rotary inertia may be taken into account to improve the classical theory. It is so called Timoshenko beam theory. These two theories are linear ones, so that it can be considered a well-defined one. In practice, however, the nonlinearities in a beam vibration arise due to the following:

1. Large curvature.
2. Longitudinal elastic forces generated due to immovable supports.
3. Longitudinal inertia forces.
4. Rotary inertia forces.

Much of the earlier work on the nonlinear transverse vibration of beams: Woinowsky-Krieger [1], Burgreen [2], MacDonald [3], Wah [4], Srinivasan [5], Evensen [6], Ray & Bert [7], Elsley [8], [9], and Dickey [10], has been considered with simply supported or clamped beams whose ends are

* Member, Seoul National University, College of Engineering

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restrained from axial displacement. In these works, the effects of longitudinal and rotary inertia, and that of shear deformation, have been neglected, and the only nonlinearity considered is that due to the average longitudinal elastic force generated due to the average midplane stretching induced when the supports are held a constant end distance. The resulting equations, involving only one nonlinear elastic term have been solved in terms of Jacobi elliptic function, or by Ritz-Galerkin method, or by the perturbation method of strained parameter. Buchanan et al. [11] derived the static equations governing the nonlinear behavior of hinged elastic bar involving the effects of large rotation, longitudinal elastic force and shear deformation. They solved it by the method of strained parameter and pointed out that for some combinations of length and cross-sectional shapes including the shear deformation causes a distinct difference in the static deflection curve. Atluri [12] studied the nonlinear inertia effects including the effects of large curvature, longitudinal inertia and rotary inertia, while the effects of midplane stretching and shear deformation are neglected. He solved the resulting nonlinear partial differential equation analytically by the perturbation method of multiple scales and concluded that the predominant nonlinearity is that due to nonlinear longitudinal inertia which is of softening type. Rao et al. [13] showed the effects of shear deformation and rotary inertia on the large-amplitude vibrations of beams by finite element method.

In the present analysis, the beam is considered as simply supported and one end of the beam ($x=l$) is assumed to be free to move in the axial direction. The effects of large curvature, longitudinal inertia and shear deformation are included, while the effects of midplane stretching and rotary inertia are ignored. The reason for omitting the rotary inertia instead of shear deformation is that the effect of latter is more predominant than that of first on the vibration frequency [14]. Analytical results by the perturbation method of multiple scales are obtained for the general response and amplitude-frequency relations. An example and comparison of results are presented.

2. Basic Equations

The equations of the lateral vibration of the beam can be written as

$$\rho\frac{\partial^2 w}{\partial t^2}=-\frac{\partial Q}{\partial x}-\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right) \tag{2.1}$$

$$\rho\frac{\partial^2 u}{\partial t^2}=-\frac{\partial N}{\partial x} \tag{2.2}$$

$$\frac{\partial M}{\partial x}=Q \tag{2.3}$$

where

x : axial coordinate along beam,
u : axial displacement of beam,
w : transverse displacement of beam,
ρ : mass per unit length,
Q : transverse shear resultant,
N : axial force positive in compression,
M : bending moment along beam.

The axial displacement due to large transverse displacement is given by

$$u(x,t)=-\frac{1}{2}\int_0^x\left(\frac{\partial w}{\partial x}\right)^2 dx \tag{2.4}$$

if in the binomial expansion only the first two terms are retained.

Substituting (2.4) into (2.2) and using the conditions $u(0,t)=0$, $N(l,t)=0$, the integration of (2.2) leads to

$$N(x,t)=-\frac{1}{2}\rho\int_x^l\frac{\partial^2}{\partial t^2}\left[\int_0^x\left(\frac{\partial w}{\partial x}\right)^2dx\right]dx. \quad (2.5)$$

Noting the relations

$$M=EI\frac{\partial\Psi}{\partial x},\quad Q=kGA\left(\Psi-\frac{\partial w}{\partial x}\right) \quad (2.6)$$

where

l : length of beam,
E : Young's modulus,
I : area moment of inertia,
Ψ : angle of rotation due to bending,
k : shear constant,
G : shear modulus,
A : cross sectional area

and eliminating Ψ from (2.1) and (2.3), we obtain

$$EI\frac{\partial^4 w}{\partial x^4}-\frac{\rho EI}{kGA}\frac{\partial^4 w}{\partial x^2\partial t^2}+\rho\frac{\partial^2 w}{\partial t^2}$$
$$=-\frac{\partial}{\partial x}\left(N\frac{\partial w}{\partial x}\right)+\frac{EI}{kGA}\frac{\partial^3}{\partial x^3}\left(N\frac{\partial w}{\partial x}\right) \quad (2.7)$$

which is a nonlinear partial differential equation because of the right-hand-side terms.

A modal expansion for w can be assumed as

$$w=\sum_{n=1}^{s}\bar{q}_n(t)\sin\frac{n\pi}{l}x \quad (2.8)$$

which satisfies the simply supported boundary conditions

$$w=\frac{\partial^2 w}{\partial x^2}=0\quad\text{at } x=0 \text{ and } x=l. \quad (2.9)$$

Substituting (2.8) into (2.9), we obtain a single nonlinear ordinary differential equation for the dependent variables $\bar{q}_n(t)$ ($n=1,2,\ldots\ldots,s$) by using the Galerkin technique.

$$\ddot{\bar{q}}_j+\omega_j^2\bar{q}_j=-\sum_m\sum_n\sum_r A_{mnrj}(q_r\dot{q}_n\dot{q}_m+q_rq_n\ddot{q}_m) \quad (2.10)$$

where

$$\omega_j^2=\frac{\omega_{j0}^2}{1+\frac{EI}{kGA}\left(\frac{j\pi}{l}\right)^2},\quad \omega_{j0}^2=\frac{j^4\pi^4}{l^4}\frac{EI}{\rho} \quad (2.11)$$

$$C_j=\frac{2\frac{\pi^3}{l^4}}{1+\frac{EI}{kGA}\left(\frac{j\pi}{l}\right)^2} \quad (2.12)$$

$$A_{mnrj}=\left(a_{mnrj}-\frac{EI}{kGA}b_{mnrj}\right)C_j \quad (2.13)$$

$$a_{mnrj}=mnr\int_0^l\left(f_{mn}\cos\frac{r\pi}{l}x+\frac{r\pi}{l}F_{mn}\sin\frac{r\pi}{l}x\right)\sin\frac{j\pi x}{l}dx \quad (2.14)$$

$$b_{mnrj}=mnr\int_0^l\left(f_{mn}''\cos\frac{r\pi}{l}x-\frac{3\pi}{l}rf_{mn}'\sin\frac{r\pi}{l}x-\frac{3\pi^2}{l^2}r^2f_{mn}\cos\frac{r\pi}{l}x-\frac{\pi^3}{l^3}r^3F_{mn}\sin\frac{r\pi}{l}x\right)\sin\frac{j\pi x}{l}dx \quad (2.15)$$

$$f_{mn}(x)=\int_0^x\cos\frac{m\pi}{l}x\cos\frac{n\pi}{l}xdx=\int_0^x f_{mn}'(x)dx \quad (2.16)$$

$$F_{mn}(x)=\int_x^l f_{mn}(x)dx=\int_x^l\left[\int_0^x\cos\frac{m\pi}{l}x\cos\frac{n\pi}{l}x\,dx\right]dx \quad (2.17)$$

3. Analytical Solution

The system of (2.10) can now be solved for any initial conditions of the type

$$\bar{q}_j(0)=\varepsilon^{\frac{1}{2}}c_j,\quad \frac{\partial q_j}{\partial t}=\varepsilon^{\frac{1}{2}}d_j\quad\text{at } t=0 \quad (3.1)$$

where ε is a small parameter that defines the amplitudes of the initial conditions. Then we may take the form for the solution of (2.10) as

$$\bar{q}_j(t)=\varepsilon^{\frac{1}{2}}q_j(t) \quad (3.2)$$

Using (3.2), equation (2.10) is reduced to

$$\ddot{q}_j+\omega_j^2 q_j=-\varepsilon\sum\sum\sum A_{mnrj}\, q_r\dot{q}_n\dot{q}_m -\varepsilon\sum\sum\sum A_{mnrj}\, q_r q_n\ddot{q}_m \qquad (3.3)$$

This type of equation can be solved by using the perturbation method of multiple scales [15]. To do this we assume that there exists a uniformly valid asymptotic solution $q_j(t)$ of the form

$$q_j(t)=\sum_{m=0}^{M-1}\varepsilon^m q_{jm}\ (T_0, T_1, \ldots, T_M)+0(\varepsilon T_M) \qquad (3.4)$$

where

$$T_m=\varepsilon^m t. \qquad (3.5)$$

The time derivatives are transformed according to

$$\frac{d}{dt}=\sum_{m=0}^{M}\varepsilon^m\frac{\partial}{\partial T_m}.$$

$$\frac{d^2}{dt^2}=\sum_m\sum_n\varepsilon^m\varepsilon^n\frac{\partial^2}{\partial T_m\partial T_n}. \qquad (3.6)$$

Substituting (3.5) and (3.6) into (3.3), and equating the coefficients of like powers of ε, we obtain the following system of equations for q_{jm} $(m=0,1,2\cdots\cdots M)$.

$$\varepsilon^0:\ \frac{\partial^2 q_{j0}}{\partial T_0^2}+\omega_j^2 q_{j0}=0, \qquad (3.7)$$

$$\varepsilon^1:\ \frac{\partial^2 q_{j1}}{\partial T_0^2}+\omega_j^2 q_{j1}=-2\frac{\partial^2 q_{j0}}{\partial T_0\partial T_1} -\sum_m\sum_n\sum_r A_{mnrj}\, q_{r0}\frac{\partial q_{n0}}{\partial T_0}\frac{\partial q_{m0}}{\partial T_0} -\sum_m\sum_n\sum_r A_{mnrj}\, q_{r0}q_{n0}\frac{\partial^2 q_{m0}}{\partial T_0^2}. \qquad (3.8)$$

Similar equations can be written for higher powers of ε.

The initial conditions (3.1) can be written as

$$q_{j0}=c_j,\quad \frac{\partial q_{j0}}{\partial T_0}=d_j \qquad \text{for}\ T_m=0, \qquad (3.9)$$

$$q_{jm}=0,\quad \frac{\partial q_{j0}}{\partial T_1}+\frac{\partial q_{j1}}{\partial T_0}=0 \qquad \text{for}\ T_m=0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (3.10)$$

The general solution of (3.7) is

$$q_{j0}=A_j(T_1, T_2, \cdots, T_M)\exp\ (i\omega_j T_0) +\bar{A}_j(T_1, T_2, \cdots, T_M)\exp\ (-i\omega_j T_0). \qquad (3.11)$$

Where $i=\sqrt{-1}$. A_j is a complex number and $\bar{A}_j$ is the complex conjugate of A_j. A_j and $\bar{A}_j$ can be determined from the initial conditions of (3.9).

Substitution for q_{j0}, equation (3.11), into (3.8), we obtain

$$\frac{\partial^2 q_{j1}}{\partial T_0^2}+\omega_j^2 q_{j1}=-2i\omega_j\left[\frac{\partial A_j}{\partial T_1}\exp\ (i\omega_j T_0) -\frac{\partial\bar{A}_j}{\partial T_1}\exp\ (-i\omega_j T_0)\right]$$
$$+\sum_m\sum_n\sum_r A_{mnrj}\omega_m\omega_n\ [A_r\exp(i\omega_r T_0) +\bar{A}_r\exp(-i\omega_r T_0)]\times A_n\exp(i\omega_n T_0) -\bar{A}_n\exp(-i\omega_n T_0)\ [A_m\exp(i\omega_m T_0) -\bar{A}_m\exp(-i\omega_m T_0)]$$
$$+\sum_m\sum_n\sum_r A_{mnrj}\omega_m^2\ [A_r\exp(i\omega_r T_0) +\bar{A}_r\exp(-i\omega_r T_0)]\times A_n\exp(i\omega_n T_0) +\bar{A}_n\exp(-i\omega_n T_0)[A_m\exp(i\omega_m T_0) +\bar{A}_m\exp(-i\omega_m T_0)] \qquad (3.12)$$

Collecting the terms on the right-hand side which vary with ω_j only, the equation can be written as

$$\frac{\partial^2 q_{j1}}{\partial T_0^2}+\omega_j^2 q_{j1}=-\left(2i\omega_j\frac{\partial A_j}{\partial T_1} +A_j\sum_{mq} g_m A_m\bar{A}_m\right)\exp(i\omega_j T_0) +\left(2i\omega_j\frac{\partial\bar{A}_j}{\partial T_1}+\bar{A}_j\sum_m g_m A_m\bar{A}_m\right)\exp (-i\omega_j T_0)+\sum_s \overline{P}_s\exp(i\omega_s T_0) +\sum_s \overline{P}_s\exp(-i\omega_s T_0) \qquad (3.13)$$

where ω_r stands for the combinations

$$\omega_s=\pm\omega_r\pm\omega_m\pm\omega_n$$

such that $\omega_s\neq\omega_j$, and

$$g_m=\begin{cases} -2A_{jjjj}\ \omega_j^2, & \text{for}\ m=j\\ -2(A_{jmjm}+A_{jmmj})\omega_m^2 & \text{for}\ m\neq j\end{cases} \qquad (3.15)$$

The explicit expressions for P_s are lengthy and hence are not recorded here. In order to eliminate the terms which produce secular terms, we must take

$$2i\omega_j \frac{\partial A_j}{\partial T_1}+A_j\sum_m g_m A_m \bar{A}_m=0. \quad (3.16)$$

The system of (3.16) can easily be solved by letting

$$A_n=\Psi_n \exp(i\theta_n), \quad (3.17)$$

where Ψ_n is a real quantity. Substituting (3.17) into (3.16), and separating real and imaginary parts, we get

$$\frac{\partial \Psi_j}{\partial T_1}=0, \quad \frac{\partial \theta_j}{\partial T_1}=\frac{1}{2\omega_j}\sum_m g_m \Psi_m^2. \quad (3.18)$$

Hence, we get

$$\Psi_j=\Psi_j(T_2, T_3, \cdots\cdots, T_M), \quad (3.19)$$

and

$$\theta_j=\frac{1}{2\omega_j}\left[\sum_m g_m \Psi_m^2\right] T_1+\theta_{j0}(T_2, \cdots\cdots, T_M). \quad (3.20)$$

Now (3.17) becomes

$$A_n(T_1, T_2, \cdots\cdots, T_M)=A_n' \exp\left[\frac{i}{2\omega_j}\sum_m g_m A_m'\bar{A}_m) T_1\right], \quad (3.21)$$

where

$$A_n'=\Psi_n(T_2, \cdots\cdots, T_M) \exp(i\theta_{n0}) \quad (3.22)$$

On account of (3.16), equation (3.13) is reduced to

$$\frac{\partial^2 q_{j1}}{\partial T_0^2}+\omega_j^2 q_{j1}=\sum_s P_s \exp(i\omega_s T_0)+\sum_s \bar{P}_s \exp(-i\omega_s T_0), \quad (3.23)$$

which has the solution

$$q_{j1}=B_j(T_1, \cdots\cdots T_M) \exp(i\omega_j T_0)+\bar{B}_j(T_1\cdots\cdots T_M) \exp(-i\omega_j T_0)+\sum_s \frac{P_s\exp(i\omega_s T_0)+P_s \exp(-i\omega_s T_0)}{\omega_j^2-\omega_s^2}, \quad (3.24)$$

Where B_j and $\bar{B}_j$ are determined from the initial conditions

$$q_{j1}=0, \quad \frac{\partial q_{j1}}{\partial T_0}=-\frac{\partial q_{j0}}{\partial T_1} \text{ for } T_m=0. \quad (3.25)$$

Therefore, in terms of t, $\bar{q}_{j(t)}$ has the form

$$\bar{q}_{j(t)}=\varepsilon^{1/2}[A_j' \exp(i\omega_j' t)+\bar{A}_j' \exp(-i\omega_j' t)]+\varepsilon^{3/2}[B_j \exp(i\omega_j t)+\bar{B}_j\exp(-i\omega_j t)+\sum_s \frac{P_s\exp(i\omega_s t)+\bar{P}_s\exp(-i\omega_s t)}{w_j^2-w_s^2}]+0(\varepsilon^{5/2}t), \quad (3.26)$$

where

$$\omega_j'=\omega_j\left[1+\frac{\varepsilon}{2\omega_j^2}\sum_m g_m A_m'\bar{A}_m'\right], \quad (3.27)$$

Equation (3.26) shows that there is the nonlinear coupling between the modes. On the other hand (3.27) indicates the effect of coupling bewteen modes on the frequency of natural vibration of each mode. Now a specific example to see the effects of the longitudinal inertia and shear deformation on the natural frequency is discussed.

4. Discussion

We now choose an example to illustrate the effects of large amplitude, longitudinal inertia force and shear deformation on this nonlinear behavior of beam. A beam of rectangular cross section with constant thickness (h) is considered. We assume that the beam is excited in the first mode with the initial conditions,

$$w(x,t)=0, \quad \frac{\partial w}{\partial t}=\varepsilon^{1/2}d_1\omega_1 \sin\frac{\pi}{l}x \quad \text{at } t=0 \quad (4.1)$$

Where ω_1 is the linear natural frequency of the first mode, then the corresponding initial conditions for $\bar{q}_j$ are

$$\bar{q}_j=0, \quad j=1,2\cdots\cdots$$
$$\frac{\partial \bar{q}_1}{\partial t}=\varepsilon^1\omega^{/2}d_{11} \quad (4.2)$$
$$\frac{\partial \bar{q}_j}{\partial t}=0, \quad j=2,3,\cdots, \text{ at } t=0$$

From (3.11) and (3.21), the zeroth-order solution, using (4.2), is

$$q_{10}=d_1 \sin w_1 t,\quad q_{k0}=0 \quad \text{for } k>1, \tag{4.3}$$

and

$$\omega_1'=\omega_1\left[1+\frac{\varepsilon d_1^2 g_1}{8\omega_1^2}\right] \tag{4.4}$$

From (2.11), we have

$$\omega_1^2=\frac{\omega_{10}^2}{1+\frac{EI}{kGA}\frac{\pi^2}{l^2}},\quad \omega_{10}^2=\frac{EI}{\rho}\frac{\pi^4}{l^4} \tag{4.5}$$

and from (3.15)

$$g_1=-2A_{1111}w_1^2, \tag{4.6}$$

in which, from (2.12)and (2.13), we have

$$A_{1111}=\left(a_{1111}-\frac{EI}{kGA}b_{1111}\right)C_1$$

$$C_1=\frac{2\frac{\pi^3}{l^4}}{1+\frac{EI}{kGA}\left(\frac{\pi}{l}\right)^2} \tag{4.7}$$

The integrations of (2.14)~(2.17) leads to

$$a_{1111}=\frac{\pi l^2}{12}-\frac{3l^2}{32\pi},\quad b_{1111}=-\frac{13\pi}{32}-\frac{\pi^3}{12}.$$

Hence (4.6) becomes

$$g_1=-\left[\frac{\pi l^2}{6}-\frac{3l^2}{16\pi}+\left(\frac{13}{16}\pi+\frac{\pi^3}{6}\right)\frac{EI}{kGA}\right]C_1\omega_1^2 \tag{4.8}$$

From (4.4), we finally obtain

$$\frac{\omega_1'}{\omega_1}=1+\frac{\varepsilon}{8}\left(\frac{d_1}{l}\right)^2\left\{-\frac{\pi^4}{6}+\frac{3\pi^2}{16}-\left(\frac{13\pi^2}{16}+\frac{\pi^4}{6}\right)2.5\delta^2\right\}\frac{1}{1+2.5\delta^2} \tag{4.9}$$

in which

$$\delta=\frac{h}{l}\ \text{(slenderness ratio)},$$

and it is assumed

$$\frac{E}{G}=2.5,\quad k=\frac{\pi^2}{12},\quad \frac{I}{A}=r^2=\frac{h^2}{12},$$

r: radius of gyration.

The ratio of the nonlinear frequency to the linear frequency with shear effect, equation (4.9), shows that the nonlinearity is of softening type which the frequency decreases with amplitude. To compare the tendency of softening effect in this problem with that of the case studied by Atluri[12], set $\delta=0.01$, then (4.9) becomes

$$\frac{\omega_1'}{\omega}=1-0.00018\varepsilon\left(\frac{d_1}{h}\right)^2 \tag{4.10}$$

For the case including rotary inertia but neglecting shear deformation Atluri [12] showed

$$\frac{\omega_1'}{\omega_1}=1-0.0003\varepsilon\left(\frac{d_1}{h}\right)^2,$$

$$\omega_1=\frac{w_{10}^2}{1+\frac{r^2\pi^2}{l^2}} \tag{4.11}$$

These two equations show that Atluri's case is more affected than the present case by the nonlinear longitudinal inertia effect.

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