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ON OSCILLATORY PROPERTIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Introduction

Consider the following functional differential equation

(A) $\qquad (r(t)x'(t))^{(n-1)}+f(x(g(t)),t)=0.$

For equation (A), the following conditions and notations are assumed to hold throughout the paper:

(a) $r(t)$ is continuous and positive for $t>0$ and $\int^{\infty}\frac{ds}{r(s)}=\infty$;

(b) $f(y,t)$ is continuous for $|y|<\infty, t>0$ and $yf(y,t)>0$ $(y\neq 0)$ for $t>0$;

(c) $g(t)$ is continuous for $t>0$, $g(t)\leq t$ and $\lim_{t\to\infty} g(t)=\infty$;

(d) $\rho(t)=\frac{1}{(n-2)!}\int_T^t \frac{s^{n-2}}{r(s)}\,ds$ and $R_{(m)}(t)=\frac{1}{m!}\int_T^t \frac{(t-s)^m}{r(s)}\,ds$, $m=0,1,\ldots,n-2$.

Recently, Singh[5] and Kusano and the present author [2] studied the oscillatory properties of equation (A). Here, we discuss the more general oscillatory properties of equation (A). We restrict our attention to solutions of (A) which exist on some ray $[t_0,\infty)$ and are nontrivial in every neighborhood of infinity. A solution is said to be oscillatory if it has arbitrarily large zeros, otherwise, it is said to be nonoscillatory. Equation (A) itself is called oscillatory if all of its solutions are oscillatory.

2. Oscillation theorems

First we mention the following elementary

LEMMA. *Let $x(t)>0$ be a solution of equation* (A). *Put $r(t)x'(t)=y(t)$, then, for n even, there exists an even integer $l(0\leq l\leq n-2)$ such that*

(i) $y(t)^{(j)}\geq 0$ *for* $j=0,1,\ldots,l$, $(-1)^{n+j-2}y(t)^{(j)}\geq 0$ *for* $j=l+1,\ldots,n-1$

for n odd, there exists an odd integer $l(0\leq l\leq n-2)$ such that

(ii) $-(1)\,y(t)^{(j)}\geq 0$ *for* $j=0,1,\cdots\cdots l$, $(-1)^{n+j-2}y(t)^{(j)}\geq 0$ *for* $j=l+1,\ldots,n-1$, *or*

(ii)—(2) $(-1)^{j+1}y(t)^{(j)}\geq 0$ *for* $j=0, 1, \cdots\cdots, n-1$.

Finally, the following inequality holds for n even or odd,

(iii) $x(t)\leq A\rho(t)$, *where A is a constant.*

PROOF. From equation (A), we have $y(t)^{(n-1)}\leq 0$, so that it follows that $y(t)\geq 0$ or $y(t)\leq 0$ eventually. For n even, we have $y(t)\geq 0$. If we suppose that $y(t)<0$, then, by Kiguradze's Lemma [1] or by a simple computation, we have that $y'(t)\leq 0$. This leads to $r(t)x'(t)=y(t)\leq y(T)<0$, so that we obtain

$$x(t)\leq x(T)+y(T)\int_T^t \frac{1}{r(s)}\,ds\to-\infty, \text{ as } t\to\infty.$$

This is a contradiction to $x(t)$ being positive. For n odd, if $y(t)\geqq 0$, then it reduces to the above case, if $y(t)<0$, then it follows that (ii)-(2). By repeating integration of $y(t)^{(n-1)}\leq 0$, we have that

$$r(t)x'(t)=y(t)\leq ct^{n-2}, \text{ where } c \text{ is a constant.}$$

From this we obtain

$$x(t)\leq A\int_T^t \frac{s^{n-2}}{r(s)}\,ds=A\rho(t), \text{ where } A \text{ is a constant.}$$

REMARK. Analogous statements of Lemma hold for $x(t)<0$.

THEOREM 1. *Suppose that*

(e) $f(x, t)/x$ *is nonincreasing for all* $x>0$ *and* $t>0$,

(f) *there is an* $\varepsilon>0$ *such that* $x^{-1+\varepsilon}f(x, t)$ *is nonincreasing in* x, *and*

(g) $R_{(2)}(t)\geq M\rho(t)$, *for some positive constant M.*

Finally, assume that

$$\int^{\infty} f(c\rho(g(s)), s)ds=\infty \text{ for any constant } c.$$

Then, every solution $x(t)$ *of* (A) *is oscillatory for n even and is oscillatory or* $\lim_{t\to\infty} x(t)=0$ *for n odd.*

PROOF. (The case n even). We assume the existence of a nonoscillatory solution $x(t)\neq 0$ of equation (A). Without loss of generality, we can assume that $x(t)>0$ eventually. Hence, we consider the following two cases:

Case (i), $(r(t)x'(t))'\geq 0$;

By Kiguradze's Lemma [1], we have that

$r(t)x'(t)\geq B_1 t^{n-2}(r(t)x'(t))^{(u-2)}$, for some positive constant B_1.

By integration of the above inequality from t_0 to t, we obtain

(1) $x(t)\geq B\rho(t)(r(t)x'(t))^{(u-2)}$, for some $B>0$.

Case (ii), $(r(t)x'(t))'\leq 0$:

By integration of (A) multiplied by $R_2(t)$, we have

(2) $(r(t)x'(t))^{(n-2)}R_{(2)}(t)-(r(t)x'(t))^{(n-3)}R_{(n-3)}(t)+\cdots\cdots$

$+\ (-1)^{n-1}(r(t)x'(t))'R_{(1)}(t)+(-1)^{n}r(t)x'(t)R_{(0)}(t)+(-1)^{n+1}x(t)$

$+\int_T^t R_{(2)}(s)f(x(g(s)),s)ds=K$, where K is a constant.

From (2), we obtain

(3) $x(t)\geq(r(t)x'(t))^{(n-2)}R_{(2)}(t)\geqq c(r(t)x'(t))^{(n-2)}\rho(t)$, for some $c>0$.

From (1) and (3), we have

(4) $x(g(t))\geq D\rho(g(t))(r(t)x'(t))^{(n-2)}$, where $D=\min(c, M)$.

By Lemma, (4), (c) and (f), we get

(5) $(-((r(t)x'(t))^{(n-2)}))'=\varepsilon((r(t)x'(t))^{(n-2)})^{-1+\varepsilon}f(x(g(t)),t)$

$=\varepsilon((r(t)x'(t))^{(n-2)})^{\varepsilon-1}x(g(t))^{1-\varepsilon}x(g(t))^{-1+\varepsilon}f(x(g(t)),t)$

$\geq\varepsilon(r(t)x'(t)^{(n-2)})^{-1+\varepsilon}(D\rho(g(t)((r(t)x'(t))^{(n-2)})^{1-\varepsilon}(A\rho(g(t)))^{-1+\varepsilon}f(A\rho(g(t)),t)$

$=Kf(A\rho(g(t),t)$, where K is a constant.

By integration of (5) from t_0 to t, we have

$$((r(t_0)x'(t_0))^{(n-2)})^{\varepsilon}-((r(t)x'(t))^{(n-2)})^{\varepsilon}\geq K\int_{t_0}^{t}f(A\rho(g(s)),s)ds$$

which leads to a contradiction that

(6) $$\int_{t_0}^{\infty}f(A\rho(g(s)),s)ds<\infty.$$

(The case n odd). If the case of Lemma (i), we have a contradiction (6) as same argument of the above one. If the case of Lemma (ii), then, from (2), (g), the boundedness of $x(t)$ and $\lim_{t\to\infty}x(t)\neq 0$, we obtain

$$\infty>\int^{\infty}R_{(2)}(s)x(g(s))\frac{f(x(g(s)),s)}{x(g(s))}ds=\int^{\infty}MKf(A\rho(g(s)),s)ds.$$

which is also a contradiction.

THEOREM 2. *Suppose that* (e) *or*

(h) $f(x, t)$ *is nondecreasing in* x *for all* $x>0$ *and* $t>0$.

Then, a necessary and sufficient condition for (A) *to have a nonoscillatory solution which is asymptotic to* $a\rho(t)$, $a\neq 0$, *as* $\to \infty$ *is that*

$$\int^{\infty} f(c\rho(g(s)), s)ds<\infty \text{ for some } c\neq 0. \tag{7}$$

PROOF. (Necessity). We may assume that $x(t)$ be a positive solution of (A) which satisfies

(8) $c_1\rho(g(t))\leq x(t)\leq c_2\rho(g(t))$, for some constants c_1, c_2 and for $t\geq t_0$.

By integrating (A) from t_0 to t we have

$$(r(t))x'(t))^{(n-2)}-(r(t_0)x'(t_0))^{(n-2)}+\int_{t_0}^{t} f(x(g(s)), s)ds=0, \tag{9}$$

which leads to

$$\int_{t_0}^{\infty}(f(x(g(s)), s)ds<\infty.$$

From this and (8), it follows that

$$\int_{t_0}^{\infty} f(c_1\rho(g(s)), s)ds<\infty \text{ in case (h)}$$

and

$$\frac{c_1}{c_2}\int_{t_0}^{\infty} f(c_2\rho(g(s)), s)ds<\infty \text{ in case (e).}$$

(Sufficiency). Put $a=c/2$ or $a=c$ according as in case (h) or in case (e) respectively and consider the integral equation

$$x(t)=a\rho(t)+\int_{T}^{t}\frac{1}{r(s_n)}\int_{T}^{s_n}\cdots\cdots\int_{T}^{s_3}\left(\int_{s_2}^{\infty} f(x(g(s_1)), s_1)ds_1\right)ds_2\cdots\cdots ds_n, \tag{10}$$

where T is chosen so large that

$$\int_{T}^{\infty} f(c\rho(g(s)), s)ds<a<\infty. \tag{11}$$

It is clear that a solution of (10) is a solution of equation (A). Let $\tau=\inf\{g(t)|$

$t\geq T\}$ and denote by $C_\rho[\tau,\infty)$ the linear space of all continuous functions x: $[\tau,\infty)\to R$ such that $\sup\{\rho(t)^{-2}|x(t)|: t\geq\tau\}<\infty$.

If we define $\|x\|_\rho=\sup\{\rho(t)^{-2}|x(t)|: t\geq\tau\}$, $x\in C_\rho[\tau,\infty)$, then we can see easily that $x\to\|x\|_\rho$ is a norm with which $C_\rho[\tau,\infty)=Y$ is a Banach space. Consider the subset X of Y satisfying

(12) $X=\{x\in Y\,|\,a\rho(t)\leq x(t)\leq 2a\rho(t),\ t\geq\tau\}$.

Clearly, X is bounded, closed and convex subset of Y. Define the operator Φ such that

(13) $$(\Phi x)(t)=\begin{cases}a\rho(t), & \tau\leq t\leq T,\\ a\rho(t)+\int_T^t\frac{1}{r(s_n)}\int_T^{s_n}\cdots\int_T^{s_3}\Big(\int_{s_2}^{\infty}f(x(g(s_1)),s_1)ds_1\Big)ds_2\cdots ds_n, & t\geq T.\end{cases}$$

Clearly, Φ is well-defined on X. We shall show that Φ is continuous and maps X into a compact subset of X.

i) Φ maps X into X. If $x\in X$, then $(\Phi v)(t)\geq a\rho(t)$, $t\leq\tau$ and

$$(\Phi x)(t)\leq a\rho(t)+\int_T^t\frac{1}{r(s_n)}\int_T^{s_n}\cdots\int_T^{n_3}(a)ds_2\cdots ds_n$$

$$\leq a\rho(t)+\int_T^t\frac{as_n^{\,n-2n}}{r(s_n)(n-2)!}ds_n\leq 2a\rho(t),\ \text{ for } t\leq\tau.$$

ii) Φ is continuous. Let $\{x_n\}\subset X$ be a convergent sequence to x: $\lim_{n\to\infty}\|x_n-x\|=0$. Since X is closed, $x\in X$. By the defintion of Φ, we see that

(14) $$|(\Phi x_n)(t)-(\Phi x)(t)|\leq\int_T^t\frac{1}{r(s_n)}\int_T^{s_n}\cdots\int_T^{s_3}\Big(\int_{s_2}^{\infty}G_n(s_1)ds_1\Big)ds_2\cdots ds_n,$$

where

(15) $G_n(s_1)=\lambda f(x_n(g(s_1)),s_1)-f(x(g(s_1)),s_1)|\leq 4f(c\rho(s_1),s_1)$.

By (14), (15) and (7), we have

(16) $$|(\Phi x_n)(t)-(\Phi x)(t)|\leq\Big(\int_T^{\infty}G_n(s_1)ds_1\Big)\Big(\int_T^t\frac{1}{r(s_n)}\int_T^{s_n}\cdots\int_T^{s_3}ds_2\cdots ds_n\Big)$$

$$\leq\Big(\int_T^{\infty}G_n(s_1)ds_1\Big)\Big(\int_T^t\frac{s_n^{\,n-2}}{r(s_n)(n-2)!}ds_n\Big)=\rho(t)\int_T^{\infty}G_n(s_1)ds_1.$$

Hence, we find

(17) $\|\Phi x_n - \Phi x\|_\rho \leq \sup_{t\geq\tau}\rho(t)^{-2}\rho(t)\int_T^\infty G_n(s_1)ds_1 \leq \rho(\tau)^{-1}\int_T^\infty G_n(s_1)ds_1.$

Observing that $\lim_{n\to\infty} G_n(s_1)=0$, which is a consequence of the convergence $x_n \to x$ in $C_\rho[\tau, \infty)$, and (15) and (7), we conclude from the Lebesque dominated convergence theorem that $\lim_{n\to\infty}\int_T^\infty G_n(s_1)ds_1=0$. Consequently, from (17) $\lim_{n\to\infty} \|\Phi x_n - \Phi x\|_\rho=0$, proving the continuity of Φ.

iii) Φ is compact. According to a theorem of Levitan [3] it suffices to show that, for any given $\varepsilon>0$, the interval $[\tau, \infty)$ can be devided into a finite number of subintervals in such a way that the oscillations on each subinterval of all functions $\rho\Phi x$, $x\in X$ are less then ε.

The first, we examine the behavior of $\rho\Phi x$ on the interval $[T, \infty)$. It holds that if $t_2>t_1>T$, then

(18) $$(\rho^{-2}\Phi x)(t_2)-(\rho^{-2}\Phi x)(t_1)=a(\rho(t_2)^{-1}-\rho(t_1)^{-1})$$
$$+\rho(t_2)^{-2}\int_T^{t_2}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3}\Big(\int_{s_2}^\infty f(x(g(s_1)), s_1), ds_1\Big)ds_2 \cdots\cdots ds_n$$
$$-\rho(t_1)^{-2}\int_T^{t_1}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3}\Big(\int_{s_2}^\infty f(x(g(s_1), s_1)ds_1\Big)ds_2\cdots\cdots ds_n.$$

It follows that

(19) $$|(\rho^{-2}\Phi x)(t_2)-(\rho^{-2}\Phi x)(t_1)|\leq 2a\rho(t_1)^{-1}$$
$$+\rho(t_2)^{-2}\Big(\int_T^\infty f(c\rho(g(s_1)), s_1)ds_1\Big)\int_T^{t_2}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3} ds_2\cdots\cdots ds_n$$
$$+\rho(t_1)^{-2}\Big(\int_T^\infty f(c\rho(g(s_1)), s_1)ds_1\Big)\int_T^{t_2}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3} ds_2\cdots\cdots ds_n$$
$$=2a\rho(t_1)^{-1}+(\rho(t_2)^{-2}\rho(t_2)+\rho(t_1)^{-2}\rho(t_1))\int_T^\infty f(c\rho(g(s_1)), s_1)ds_1$$
$$\leq 4a\rho(t_1)^{-1}.$$

Noting that $\rho(t_1)^{-1}\to 0$ as $t_1\to\infty$, we conclude from (19) that there exists $t^*>T$ such that for all $x\in X$,

$$|(\rho^{-2}\Phi x)(t_2)-(\rho^{-2}\Phi x)(t_1)|<\varepsilon \text{ if } t_2>t_1>t^*,$$

so that the oscillations of all $\rho^{-2}\Phi$ on $[t^*, \infty)$ are less than ε. Let $T \leq t_1 \leq t_2 \leq t^*$, then from (18), we obtain

$$(20) \quad |(\rho^{-2}\Phi x)(t_2)-(\rho^{-2}\Phi x)(t_1)| \leq a|\rho(t_2)^{-1}-\rho(t_1)^{-1}|$$
$$+\rho(t_2)^{-2}\int_{t_1}^{t_2}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3}\Big(\int_{s_2}^{\infty}f(x(g(s_1)), s_1)ds_1\Big)ds_2\cdots\cdots ds_n$$
$$+|\rho(t_2)^{-2}-\rho(t_1)^{-2}|\int_T^{t_1}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3}\Big(\int_{s_2}^{\infty}f(x(g(s_1)), s_1)ds_1\Big)ds_2\cdots\cdots ds_n$$
$$\leq |a\rho(t_2)^{-1}-\rho(_1)^{-1}|+\rho(T)^{-2}\Big(\int_T^{\infty}f(c\rho(g(s_1)), s_1)ds_1\Big)\Big(\int_{t_1}^{t_2}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\cdots\int_T^{s_3}ds_2$$
$$\cdots ds_n\Big)+|\rho(t_2)^{-2}-\rho(t_1)^{-2}|\Big(\int_T^{\infty}f(c\rho(g(s_1)), s_1)ds_1\Big)\Big(\int_T^{t^*}\frac{1}{r(s_n)}\int_T^{s_n}\cdots\int_T^{s_3}ds_2\cdots ds_n\Big).$$

This inequality ensures the existence of $\delta>0$ such that for all $x\in X$

$$|(\rho^{-2}\Phi x)(t_2)-(\rho^{-2}\Phi x)(t_1)|<\varepsilon \text{ if } t_2-t_1<\delta.$$

A similar argument holds regarding the oscillations of $\rho^{-2}\Phi x$, on $[\tau, T]$. Hence, it follows that the whole interval $[\tau, \infty)$ admits a decomposition into a finite number of subintervals on each of which all functions $\rho^{-2}\Phi x$, $x\in X$, have oscillations less than ε. So that by applying Schauder's fixed point theorem, we have an $x\in X$ of a fixed point of Φ. Then, by the definition of Φ, $x(t)$ is a solution of equation (10). So that, we can see easily that $x(t)$ satisfies

$$(21) \qquad \lim_{t\to\infty}(x(t)/\rho(t))=a.$$

Hence, $x(t)$ is a solution of equation (A), which tends to $a\rho(t)$.

THEOREM 3. *Suppose that* (e) *or* (h). *Then a necessary and sufficient condition for* (A) *to have a nonoscillatory solution which is asymptotic to* $a(\neq 0)$, *as* $t\to\infty$ *is that*

$$(22) \qquad \int^{\infty}R_2(s)f(c, s)ds\to\infty, \text{ for some } c>0.$$

PROOF. (Necessity). We may assume that $x(t)$ be a positive solution of (A) which satisfies that

(23) $c_1 \leq x(t) \leq c_2$, for some positive constants c_1 and c_2 nad for $t \geq T$. For n even, from Lemma, we have $r(t)x'(t)\geq 0$ and $(r(t)x'(t))'\leq 0$.

If not, then we have

$$x(t)\geq x(T)+r(T)x'(T)\int_T^t \frac{ds}{r(s)},$$

which leads to a contradiction to (23). So that we have

(24) $$(-1)^j(r(t)x'(t))^{(j)}\geq 0,\ j=0,\ 1,\ \cdots\cdots,\ n-1.$$

For n odd, from Lemma and the same argument of the above one, we have (24) or

(25) $$(-1)^{j+1}(r(s)x'(t))^{(j)}\geq 0,\ j=0,\ 1,\ \cdots\cdots,\ n-1.$$

By intergration of (A) multiplied by $R_2(t)$, we obtain (2) for n even or odd. So that, by considering of (24), (25) and (2), we obtain

$$\int_T^\infty R_2(s)f(x(g(s)), s)ds<\infty.$$

From this and (23), we have that

$$\int_T^\infty R_2(s)f(c_1, s)ds<\infty \text{ in case (h)}$$

and

$$\frac{c_1}{c_2}\int_T^\infty R_2(s)f(c_2, s)ds<\infty \text{ in case (e).}$$

(Sufficiency). Put $a=c/2$ or $a=c$ according as in case (h) or in case (e) respectively and $d=2a$ if n even and $d=a$ if n odd. Consider the integral equation

(26) $$x(t)=d+(-1)^{n+1}\int_t^\infty \frac{1}{r(s)}\int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} f(x(g(u)), u)duds.$$

It is clear that a solution of (26) is a solution of equation (A). Let T be chosen so large that

(27) $$\int_T^\infty R_2(s)f(c, s)ds<a.$$

Let $\tau=\inf\ \{g(t)\,|\,t\geq T\}$ and denote by $C[\tau, \infty)$ the linear space of all continuous x: $[\tau, \infty)\to R$ such that $\sup\{|x(t)|: t\geq\tau\}<\infty$.

If we define $\|x\|=\sup\{|x(t)|:\ t\geq\tau\}, x\in C[\tau, \infty)$, then we can easily see that $x\to\|x\|$ is a norm with which $C[\tau, \infty)$ is a Banach space. Consider the set X of all functions $x\in C[\tau, \infty)$ satisying

(28) $$a\leq x(t)\leq 2a \text{ on } [\tau, \infty).$$

Clearly, X is bounded, closed and convex subset of $C[\tau,\infty)$. Define the operator Φ suth that

$$(29)\quad (\Phi x)(t)=\begin{cases} d+(-1)^{n+1}\displaystyle\int_T^\infty \frac{1}{r(s)}\int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} f(x(g(u)),u)duds, & \text{for } \tau\le t\le T, \\ d+(-1)^{n+1}\displaystyle\int_t^\infty \frac{1}{r(s)}\int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} f(x(g(u)),u)duds, & \text{for } t\ge T. \end{cases}$$

The following inequality

$$(30)\quad \int_t^\infty \frac{1}{r(s)}\int_s^\infty \frac{(u-s)^{n-2}}{(n-2)!} f(x(g(u)),u)duds$$

$$=\int_t^\infty\left(\int_t^u \frac{(u-s)^{n-2}}{r(s)(n-2)!}ds\right)f(x(g(u)),u)du\le\int_t^\infty R_2(u)f(x(g(u)),u)du$$

shows that (29) is well-defined. We shall show that Φ is continuous and maps X into a compact subset of X.

i) Φ maps X into X. If $x\in X$, then we have

$$(31)\quad \begin{cases} d\le(\Phi x)(t)\le d+\displaystyle\int_T^\infty R_2(u)f(x(g(u)),u)du, & \text{for } n \text{ odd,} \\ \text{and} & \\ d-\displaystyle\int_T^\infty R_2(u)f(x(g(u)),u)du\le(\Phi x)(t)\le d, & \text{for } n \text{ even.} \end{cases}$$

From (31) and the definition of d, it follows that $a\le(\Phi x)(t)\le 2a$.

ii) Φ is continuous. Let $\{x_n\}\subset X$ be a convergent sequence to x: $\lim_{n\to\infty}\|x_n-x\|=0$. Since, X is closed, $x\in X$. By the definition of Φ, we see that

$$(32)\qquad |(\Phi x_n)(t)-(\Phi x)(t)|\le\int_T^\infty R_2(u)A_n(u)du,$$

where

$$(33)\quad A_n(u)=|f(x_n(g(u)),u)-f(x(g(u)),u)|\le 4f(c,u).$$

Observing that $\lim_{n\to\infty}A_n(u)=0$, which is a consequence of the convergence $x_n\to x$ in $C[\tau,\infty)$ and (33) and (27), We conclude from the Lebesgue dominated convergence theorem that $\lim_{n\to\infty}\int_T^\infty R_2(u)A_n(u)du=0$. From this, we have $\lim_{n\to\infty}\|\Phi x_n-\Phi x\|=0$, proving the continuity of Φ.

iii) We claim that Φx is a compact subset of X. Since each Φx is constant on $[\tau, T]$, we need only to examine the behavior of Φx on the interval $[T, \infty)$, An easy computation shows that if $t_2 > t_1 > T$, then

$$(34) \quad |(\Phi x)(t_2)-(\Phi x)(t_1)| \leq \int_{t_2}^{\infty} R_2(u)f(x(g(u)), u)du$$

$$+\int_{t_1}^{\infty} R_2(u)f(x(g(u)), u)du+\int_{t_1}^{t_2} R_2(u)f(x(g(u)), u)du.$$

As $t_1 \to \infty$ in (34), we have that for any $\varepsilon > 0$, there exists t^* which is independent of x, such that

$|(\Phi x)(t_2)-(\Phi x)(t_1)| < \varepsilon$, for t_2, $t_1 \geq t^*$ and any $x \in X$.

Let $T \leq t_1 \leq t_2 \leq t^*$, then we have

$$(35) \quad |(\Phi x)(t_2)-(\Phi x)(t_1)| \leq \left|\int_{t_2}^{\infty} R_2(u)f(x(g(u)), u)du-\int_{t_1}^{\infty} R_2(u)f(x(g(u)), u)du\right|$$

$$=\left|\int_{t_1}^{t_2} R_2(u)f(x(g(u)), u)du\right|.$$

This inequality (35) ensures the existence of $\delta > 0$ such that for all $x \in X$ $|(\Phi x)(t_2)-(\Phi x)(t_1)| < \varepsilon$, if $t_2 - t_1 < \delta$. Hence, it follows, that the whole interval $[\tau, \infty)$ admits a decomposition it into a finite number of subintervals on each of which all functions Φx, $x \in X$ have oscillations less then ε.

So that by applying Schauder's fixed point theorem, we have an $x \in X$ of a fixed point of Φ. Then, by the definition of Φ, $x(t)$ is a solution of equation (26) which tends to d as $t \to \infty$.

THEOREM 4. *Suppose that* $g'(t) \geq 0$, (h) *and*

(i) *there is an* $\varepsilon > 0$ *such that* $x^{-1-\varepsilon}f(x, t)$ *is nondecreasing in* x, *hold. If* $\int^{\infty} R_2(g(u))f(c_1, u)du=\infty$ *and* $\int^{\infty} R_1(g(u))f(c_2, u)du=\infty$ *are satisfied for any* $c_1 > 0$ *and* $c_2 > 0$. *Then, every solution of* (A) *is oscillatory for n even, and oscillatory or monotonically to zero as* $t \to \infty$, *for n odd.*

PROOF. Let $x(t)$ be a nonoscillatory solution of (A) with the property $\lim_{t \to \infty} x(t) \neq 0$. Then, we may suppose $x(t) > 0$. In this case, from $(r(t)x'(t))^{(n-1)} \leq 0$, we distinguish the following two cases:

Case (i): $r(t)x'(t) \geq 0$.

If we put $y(t)=r(t)x'(t)$, then $y(t)\geq 0$, $y^{(n-1)}(t)\leq 0$. So that by Lemma, we obtain the following two cases:

(i) $y(t), y(t)', \cdots\cdots, y(t)^{(n-2)}$ converge monotonically to zero as $t\to\infty$, In this case we have

$$(36)\quad (-1)^n r(g(t))x'(g(t))\geq\int_t^\infty \frac{(g(s)-g(t))^{n-2}}{(n-2)!} f(x(g(s)), s)ds,$$

which occur only in the case for n even. By (36), we have

$$(37)\quad x'(g(t))\geq(1/r(g(t)))^{-1}\int_t^\infty \frac{(g(s)-g(t))^{n-2}}{(n-2)!} f(x(g(s)), s)ds.$$

By integration of (37) multiplied by $g'(t)$, we obtain

$$(38)\quad x(g(t))\geq\int_T^t \frac{g'(t)}{r(g(s))}\int_s^t \frac{(g(u)-g(s))^{n-2}}{(n-2)!} f(x(g(u)), u)duds$$

$$=\int_T^t\int_{g(T)}^{g(u)} \frac{(g(u)-v)^{n-2}}{(n-2)!r(v)} dvf(x(g(u)), u)du$$

$$=\int_T^t R_2(g(u))f(x(g(u)), u)du.$$

Since $x'(t)\geq 0$, we have

(39) $x(g(t))\geq c>0$, where c is a constant.

By the assumptions (i) and (39), we have that

$$f(x(g(t)), t)=x(g(t))^{1+\varepsilon}\ x(g(t))^{-1-\varepsilon} f(x(g(t)), t)\geq x(g(t))^{1+\varepsilon}\ c^{-1-\varepsilon} f(c, t).$$

So that we have

$$(40)\quad x(g(t))^{-1-\varepsilon}\leq\{c^{-1-\varepsilon}\int_T^t R_2(g(s))x(g(s))^{1+\varepsilon} f(c, s)ds\}^{-1-\varepsilon}.$$

By integration of (40) multiplied by $R_2(g(t))x(g(t))^{1+\varepsilon} f(c, t)$, we obtain

$$(41)\quad \int_K^t R_2(g(s))f(c, s)ds\leq c^{1+\varepsilon}\{c^{-1-\varepsilon}\int_T^s R_2(g(u))x(g(u))^{1+\varepsilon}f(c, u)du\}^{-\varepsilon}\Big|_{s=K}^{s=t},$$

where K is a large constant.

From (40), we have a contradiction that

$$(42)\qquad \int_T^\infty R_2(g(s))f(c, s)ds<\infty.$$

(II) $\lim_{t\to\infty} y^{(i)}(t)\neq 0$, for some $i\in\{0, 1, \cdots\cdots, n-2\}$.

In this case, we have

$$(43)\quad r(g(t))x'(g(t))\geq \frac{(g(t)-g(t_2))^{n-2}}{(n-2)!}\int_t^\infty f(x(g(s)), s)ds$$

and from this, we obtain

$$(44)\quad x(g(t))\geq \int_{t_2}^{t}\int_{g(t_2)}^{g(s)} \frac{(v-g(t_2))^{n-2}}{(n-2)!\,r(v)}dv f(x(g(s)), s)ds$$

$$=\int_{t_2}^{t} R_1(g(s))f(x(g(s)), s)ds.$$

From (44) and by the same argument of Case (I), we have

$$(45)\qquad \int^\infty R_1(g(s))f(c, s)ds<\infty.$$

Case ii): $r(t)x'(t)\leq 0$.

In this case, we see that the case for n odd. By integration of (A) multiplied by $R_2(t)$, we have (2), from which we obtain

$$(46)\qquad \int^\infty R_2(g(s))f(c, s)ds<\infty.$$

THEOREM 5. *Suppose that* $g'(t)\geq 0$ *and* (i). *If*

$$(47)\qquad \int^\infty R_2(g(s))f(c, s)ds=\infty \text{ for any } c>0,$$

then, every bounded solution of (A) *is oscillatory for* n *even and is oscillatory or monotonically to zero as* $t\to\infty$ *for* n *odd.*

PROOF. From the proof part of Theorem 4, we see that the Case (II) does not happen. So that, Theorem 5 is proved.

REMARKS. It is noted that our theorems contain the cases for $f(y, t)$ being sublinear or superlinear. For $n=2$, our theorems are identical to the one's of ([4], Sec. 2).

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