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SOME THEOREMS ON GENERALISED LAPLACE TRANSFORM

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1. Introduction

The integral equation

$$\phi(p)=p\int_0^\infty e^{-pt}f(t)dt, \ \ \mathrm{Re}\,p>0 \tag{1.1}$$

represents the classical Laplace transform and the functions $\phi(p)$ and $f(t)$, related by (1.1), are said to be operationally related to each other. The functions $f(t)$ and $\phi(p)$ are called the original and the image respectively. Symbolically, we write

$$\phi(p)\doteqdot f(t) \text{ or } f(t)\doteqdot\phi(p). \tag{1.2}$$

Many generalisation of this classical Laplace transform have been given from time to time by various mathematicians like Meijer [7, (i), (ii)], Boas [1], Varma [9, (i), (ii)] and others. Mainra [6, (ii)] gave the following generalisation of this transform

$$\phi(p)=p\int_0^\infty e^{-\frac{1}{2}pt}W_{k+\frac{1}{2},\ m}(pt)(pt)^{-\lambda-\frac{1}{2}}(f(t)dt, \ \ \mathrm{Re}\,p>0, \tag{1.3}$$

where $W_{k,m}$ is the Whittakar function [10].

Here also, we call $\phi(p)$, the image and $f(t)$. the original, and write symbolically

$$\phi(p)\underset{m}{\overset{k}{\lambda}}f(t) \text{ or } f(t)\underset{m}{\overset{k}{\lambda}}\phi(p). \tag{1.4}$$

If we take $\lambda=k$ in (1.3) we get Meijer's transform, $\lambda=-m$ gives Varma's transform, and finally $\lambda=k=\pm m$ gives the classical Laplace transform.

In the theorems that follow, we have assumed $f(t)\doteqdot\phi(p)$ or $f(t)\underset{m}{\overset{k}{\lambda}}\phi(p)$ and in both these cases the defining integrals (1.1) or (1.3) are assumed to be absolutely convergent. This fact has not been mentioned in the statements of the theorems.

2. THEROEM 2.1. *If* $f(t) \underset{m}{\overset{k}{\overleftarrow{\underrightarrow{\lambda}}}} \phi(p)$, *then*

$$-p^{-1-\lambda+m}\frac{d}{dp}\left\{\frac{\phi(p)}{p^{-1-\lambda+m}}\right\} \underset{m+\frac{1}{2}}{\overset{k+\frac{1}{2}}{\overleftarrow{\underrightarrow{\lambda+\frac{1}{2}}}}} tf(t), \quad \mathrm{Re}\, p>0, \tag{2.1}$$

provided $t^{-\lambda\pm m} f(t)$ *is continuous and absolutely integrable in* $(0, a)$, *where* a *is arbitrary.*

PROOF. We have $f(t) \underset{m}{\overset{k}{\overleftarrow{\underrightarrow{\lambda}}}} \phi(p)$.

Hence

$$\frac{\phi(p)}{p^{1-\lambda+m}}=\int_0^\infty \frac{e^{-\frac{1}{2}pt}W_{k+\frac{1}{2},\, m}(pt)}{(pt)^{m+1/2}} t^{m-\lambda} f(t)dt. \tag{2.2}$$

Making use of a known result [3, (i), pp. 264 and 258], we have

$$\frac{d}{dz}\left[\frac{e^{-z/2}\, W_{k,m}(z)}{z^{m+\frac{1}{2}}}\right]= -\frac{e^{-z/2}\, W_{k+\frac{1}{2},\, m+\frac{1}{2}}(z)}{z^{m+1}}. \tag{2.3}$$

Differentiating (2.2) with respect to p and making use of (2.3), we get

$$\frac{d}{dp}\left[\frac{\phi(p)}{p^{1-\lambda+m}}\right]=-\int_0^\infty \frac{e^{-\frac{1}{2}pt}\, W_{k+1,\, m+\frac{1}{2}}(pt)}{(pt)^{m+1}} t^{1+m-\lambda} f(t)dt. \tag{2.4}$$

The process of differentiation under the sign of integration is justified as the integral on the right of (2.4) is uniformly convergent for $\alpha<p<\beta$, where α and β are arbitrary positive number [2, pp. 200—201].

Now, multiplying both sides of (2.4) by $-p^{1-\lambda+m}$ the desired result follows immediately.

COROLLARY I. *Taking* $k=\lambda=m$, *we get*

"*if* $f(t)\doteqdot\phi(p)$, *then* $tf(t)\doteqdot -p\frac{d}{dp}\left\{\frac{\phi(p)}{p}\right\}$."

This is a known result [8].

COROLLARY II. *If* $t^{-\lambda+m} f(t)$ *and* $t^{-\lambda-m-n} f(t)$ *are continuous and absolutely integrable in* $(0, a)$, *where* a *is arbitrary, repeated use of the theorem* 2.1 *gives:*

"*If* $f(t)\underset{\overrightarrow{m}}{\overset{\overleftarrow{k}}{\lambda}}\phi(p)$, *then*

$$t_n f(t)\underset{\overrightarrow{m+\frac{1}{2}n}}{\overset{\overleftarrow{k+\frac{1}{2}n}}{\lambda+\frac{1}{2}n}}(-1)^n p^{1-\lambda+m}\frac{d^n}{dp^n}\left\{\frac{\phi(p)}{p^{1-\lambda+m}}\right\},\ \text{Re}\ p>0." \tag{2.5}$$

COROLLARY III. *Taking* $k=\lambda=m$ *in* (2.5), *we get*

"*if* $f(t)\risingdotseq\phi(p)$, *then* $t^n f(t)\risingdotseq(-1)^n p\dfrac{d^n}{dp^n}\left\{\dfrac{\phi(p)}{p}\right\}$."

This is again a known result [8].

THEOREM 2.2. *If* $f(t)\underset{\overrightarrow{m}}{\overset{\overleftarrow{k}}{\lambda}}\phi(p)$, *then*

$$\frac{f(t)}{t}\underset{\overrightarrow{m-\frac{1}{2}}}{\overset{\overleftarrow{k-\frac{1}{2}}}{\lambda-\frac{1}{2}}}p^{1-\lambda+m}\int_p^\infty\frac{\phi(x)}{x^{1-\lambda+m}}dx,\quad p>0, \tag{2.6}$$

provided the integral on the right of (2.6) *converges and the defining integral for* (2.6) *converges absolutely.*

PROOF. We have $f(t)\underset{\overrightarrow{m}}{\overset{\overleftarrow{k}}{\lambda}}\phi(p)$.

Hence, we can write

$$\frac{\phi(x)}{x^{1-\lambda+m}}=\int_0^\infty\frac{e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},m}(xt)}{(xt)^{m+\frac{1}{2}}}t^{m-\lambda}f(t)dt. \tag{2.7}$$

Integrating both the sides of (2.7) with respect to x from p to $\infty, p>0$, we get

$$\int_p^\infty\frac{\phi(x)}{x^{1-\lambda+m}}dx=\int_p^\infty\int_0^\infty\frac{e^{-\frac{1}{2}xt}W_{k+\frac{1}{2},m}(xt)}{(xt)^{m+1/2}}t^{m-\lambda}f(t)dtdx. \tag{2.8}$$

Changing the order of integrations on the right of (2.8), which is justified under the conditions stated in the theorem, and then evaluating the x-integral by making use of [3, (ii), pp. 411], we get

$$\int_p^\infty \frac{\phi(x)}{x^{1-\lambda+m}}\,dx=\int_0^\infty e^{-\frac{1}{2}pt}\,W_{k,m-\frac{1}{2}}(pt)(pt)^{-m}t^{m-\lambda-1}f(t)\,dt,$$

which proves the theorem.

COROLLARY I. *Taking $k=\lambda=m$, we get the known result* [8],

"*if* $f(t)\doteqdot\phi(p)$, *then* $\dfrac{f(t)}{t}\doteqdot p\displaystyle\int_p^\infty \frac{\phi(x)}{x}\,dx$."

COROLLARY II. *By repeated use of the theorem* 2.2, *we get*

"*if* $f(t)\ \overset{\overset{k}{\longleftarrow}}{\underset{\underset{m}{\longrightarrow}}{\lambda}}\ \phi(p)$, *then*

$$\frac{f(t)}{t^n}\ \overset{\overset{k-\frac{1}{2}n}{\longleftarrow}}{\underset{\underset{m-\frac{1}{2}n}{\longrightarrow}}{\lambda-\tfrac{1}{2}n}}\ p^{1+m-\lambda}\int_p^\infty\int_x^\infty\cdots\int_x^\infty \frac{\phi(x)}{x^{1-\lambda+m}}(dx)^n, \tag{2.9}$$

provided the integral on the right of (2.9) *exists and the defining integral for* (2.9) *converges absolutely.*"

COROLLARY III. *Taking $k=\lambda=m$ in* (2.9), *we get*

"*if* $f(t)\doteqdot\phi(p)$, *then* $\dfrac{f(t)}{t^n}\doteqdot p\displaystyle\int_p^\infty\int_x^\infty\cdots\int_x^\infty \frac{\phi(x)}{x}(dx)^n$."

This is again a known result [8].

THEOREM 2.3. *If* $f(t)\ \overset{\overset{k}{\longleftarrow}}{\underset{\underset{m}{\longrightarrow}}{\lambda}}\ \phi(p)$, *then*

$$\frac{f(t)}{t}\ \overset{\overset{k-\frac{1}{2}}{\longleftarrow}}{\underset{\underset{m-\frac{1}{2}}{\longrightarrow}}{\lambda-\tfrac{1}{2}}}\ \left[\frac{\Gamma(1-2m)}{\Gamma(1-k-m)}\,p^{1+m-\lambda}\int_0^\infty t^{m-\lambda-1}\,f(t)\,dt-p^{1+m-\lambda}\int_0^p \frac{\phi(p)}{x^{1+m-\lambda}}\,dx\right],\quad p>0, \tag{2.10}$$

provided the right hand side of (2.10) *exists,* $t^{m-\lambda-1}f(t)$ *is continuous and absolutely integrable in* $(0,\infty)$, *the defining integral for* (2.10) *is absolutely convergent, and* Re $m<\frac{1}{2}$.

PROOF. We have $f(t) \overset{k}{\underset{m}{\overleftrightarrow{\lambda}}} \phi(p)$.

Hence, we can write

$$\frac{\phi(x)}{x^{1-\lambda+m}}=\int_0^\infty \frac{e^{-\frac{1}{2}xt}\,W_{k+\frac{1}{2}}(xt)}{(xt)^{m+\frac{1}{2}}}\,t^{m-\lambda}f(t)dt. \tag{2.11}$$

Integrating both the sides of (2.11) with respect to x from 0 to p, we get

$$\int_0^p \frac{\phi(x)}{x^{1-\lambda+m}}dx=\int_0^p\int_0^\infty \frac{e^{-\frac{1}{2}xt}\,W_{k+\frac{1}{2},\,m}(xt)}{(xt)^{m+\frac{1}{2}}}\,t^{m-\lambda}f(t)dtdx. \tag{2.12}$$

Changing the order of integrations on the right of (2.12), which is justified under the conditions stated in the theorem, and then evaluating the x-integral by making use of [3, (ii), pp. 411 and 407], we get

$$\int_0^p \frac{\phi(x)}{x^{1-\lambda+m}}dx=\int_0^\infty t^{m-\lambda}f(t)\Big[\frac{\Gamma(1-2m)}{\Gamma(1-k-m)t}-\frac{e^{-\frac{1}{2}pt}\,W_{k,\,m-\frac{1}{2}}(pt)}{t(pt)^m}\Big]dt.$$

On interpreting, the theorem follows immediately.

COROLLARY I. *Applying the uniqueness theorem* [6, (ii)] *to theorems* 2.2 *and* 2.3, *we get*

"*if* $f(t) \overset{k}{\underset{m}{\overleftrightarrow{\lambda}}} \phi(p)$, *then*

$$\frac{\Gamma(1-2m)}{\Gamma(1-k-m)}\int_0^\infty \frac{f(t)}{t^{1+\lambda-m}}\,dt=\int_0^\infty \frac{\phi(p)}{p^{1-\lambda+m}}dp.$$

provided the integrals on both the sides exist."

COROLLARY II. *Taking* $k=\lambda=m$ *in the above corollary, we get*

"*if* $f(t)\doteqdot\phi(p)$, *then* $\int_0^\infty \frac{f(t)}{t}dt=\int_0^\infty \frac{\phi(p)}{p}dp.$"

This is a known result [8].

3. Examples

(i) Let $f(t)=t^{u}{}_{1}F_{2}\left[\begin{matrix}u-k-\lambda+1; \\ \\ u\pm m-\lambda+1;\end{matrix}\; -t\right]$, $\mathrm{Re}(1+u\pm m-\lambda)>0$.

Then

$$\phi(p)=\frac{\Gamma(u\pm m-\lambda+1)}{\Gamma(u-k-\lambda+1)}p^{-u}e^{-p^{-1}},\quad [6,\ (\mathrm{ii})].$$

Therefore, applying the theorem 2.1, we get

$$t^{u+1}{}_{1}F_{2}\left[\begin{matrix}u-k-\lambda+1; \\ \\ u\pm m-\lambda+1;\end{matrix}\; -t\right] \underset{\substack{\longrightarrow \\ m+\frac{1}{2}}}{\overset{\substack{k+\frac{1}{2} \\ \longleftarrow}}{\lambda+\frac{1}{2}}} \frac{\Gamma(u\pm m-\lambda+1)}{\Gamma(u-k-\lambda+1)}\{(u+m-\lambda+1)p-1\}p^{-u-2}e^{-p^{-1}}, \tag{3.1}$$

$\mathrm{Re}\ p>0$, $\mathrm{Re}\ (u\pm m-\lambda+1)>0$.

(ii) Let $f(t)=t^{u+v}{}_{2}F_{2}\left[\begin{matrix}v, v-\lambda-k+u+1; \\ \\ v+u\pm m-\lambda+1;\end{matrix}\; -\alpha t\right]$.

Then

$$\phi(p)=\frac{\Gamma(1+u+v-\lambda\pm m)}{\Gamma(1+v+u-\lambda-k)}p^{-u}(p+\alpha)^{-v},\quad [4].$$

$$\mathrm{Re}(1+u+v-\lambda\pm m)>0,\ \mathrm{Re}\ p>0,\ |\alpha|<|p|.$$

Evaluating the integral on the right of (2.6) by making use of [3, (ii), pp. 201] and applying the theorem 2.2, we get

$$t^{v+u-1}{}_{2}F_{2}\left[\begin{matrix}v, v-k-\lambda+u+1; \\ \\ v+u\pm m-\lambda+1;\end{matrix}\; -\alpha t\right] \underset{\substack{\longrightarrow \\ m-\frac{1}{2}}}{\overset{\substack{k-\frac{1}{2} \\ \longleftarrow}}{\lambda-\frac{1}{2}}} \frac{\Gamma(1+u+v-\lambda-m)\,\Gamma(m+u+v-\lambda)}{\Gamma(1+u+v-\lambda-k)}p^{1-v-m}{}_{2}F_{1}\left[\begin{matrix}v, v+u+m-\lambda; \\ \\ m+u+v+\lambda+1;\end{matrix}\; -\frac{\alpha}{p}\right], \tag{3.2}$$

$\mathrm{Re}(u+v-\lambda\pm m)>0,\ p>0,\ |\alpha|<|p|.$

(iii) Let $f(t)=t^{v}\tilde{w}^{2v-2k-2\lambda+1/2}_{2v+2m-2\lambda+1/2,\ 2v-2m-2\lambda+1/2}(2\sqrt{t})$.

Then

$$\phi(p)=\sqrt{2}\,e^{-\frac{1}{2p}}W_{k+\frac{1}{2},\,m}\left(\frac{1}{p}\right)p^{\lambda-2v},\quad [6,\ (\text{ii})],$$

$\mathrm{Re}\left(2v\pm 2m-2\lambda+\frac{3}{2}\right)>0.\ \ \mathrm{Re}\left(2v-2\lambda+\frac{3}{2}\right)>0,\ p>0.$

By making use of [3, (ii), pp. 407], we get

$$\int_0^\infty \frac{\phi(p)}{x^{1-\lambda+m}}dx=\frac{\sqrt{2}\,\Gamma\left(2m+2v-2\lambda+\frac{1}{2}\right)\Gamma\left((2v-2\lambda+\frac{1}{2}\right)}{\Gamma\left(2v-2\lambda+m-k+\frac{1}{2}\right)}.$$

Therefore, applying Corollary I of the theorem 2.3, we get

$$\int_0^\infty t^{v+m-\lambda-1}\,\tilde{w}^{2v-2k-2\lambda+1/2}_{2v+2m-2\lambda+1/2,\ 2v-2m-2\lambda+1/2}(2\sqrt{t})dt$$

$$=\frac{\sqrt{2}\,\Gamma\left(2v+2m-2\lambda+\frac{1}{2}\right)\Gamma\left(2v-2\lambda+\frac{1}{2}\right)\Gamma(1-k-m)}{\Gamma\left(2v+m-2\lambda-k+\frac{1}{2}\right)\Gamma(1-2m)} \tag{3.3}$$

$p>0,\ \mathrm{Re}\left(2v\pm 2m-2\lambda+\frac{1}{2}\right)>0,\ \ \mathrm{Re}\left(2v-2\lambda+\frac{1}{2}\right)>0.\ \ \mathrm{Re}(1-k-m)>0.$

This is a known result [6, (i)].

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REFERENCES

[1] Boas, R.P. (Jr.), *Generalised Laplace integrals,* Bull. Amer. Math. Soc., 48(1942).

[2] Carslaw, H.S., *Introduction to the theory of Fourier's Series and integrals,* (Dover Pub., IIIrd revised edition).

[3] Erdelyi. A., (i) *Higher Transcendental Functions,* Vol. I (1953). (Compiled by the staff of the Bateman manuscript project with Prof. Erdelyi as Director), (McGraw Hill Pub).

(ii) *Tables of Integral Transforms* Vol. II (1954), ibid.

[4] Jain, M.K., On Meijer's transform, Acta. Math., 93, (1955).

[5[Luke, Y.L., *The special Functions and their approximations*, Vol. I, and II, Academic Press, New York and London, (1969).

[6] Mainra, V.P., (i) *A new transform*, Bull. Cal. Math. Soc., 50, (1958).
(ii) *A new generalisation of the Laplace transform*, ibid, 53, (1961).

[7] Meijer, C.S., (i) *Eine neve Erweiterung der Laplace transform*, Proc. Ned. Acad. Wetersch., 44, (1941a).
(ii) *Integral darstellunger, für Whittakarsche funktionen und ihre produkte*, ibid.

[8] Memorial Des Sciences Mathematiques, Fascicule CXIII, Paris Gauthier-Villars, Imprimeur-Editeur(1950).

[9] Varma, R.S., (i) *A generalisation of Laplace transform*, Current Sci., 16, (1947).
(ii) *On a generalisation of Laplace transform*, Proc. Nat. Acad. Sci. (India), Sec. A, 20, (1951).

[10] Whittaker, E.T. and Watson, G.N., *A Course of Modern Analysis*, Camb. Univ. Press, (1940).