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A COMPOSITE INTEGRAL TRANSFORM WITH SPHEROIDAL WAVE FUNCTION AS KERNEL

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1. Introduction

Recenity Gupta [1] has introduced an integral transform applicable to spheroidal wave functions analogous to finite Fourier transform and gave an account of the simple properties of the transform as well as its application to the solution of a few boundary value problems relating to spheroids. In a series of papers [2, 3, 4], Wankhede and Bhonsle has established the Sturm-Liouville transform for composite region consisting of k-layers by considering series expansion of an arbitrary function in terms of eigen-functions of Sturm-Liouville linear homogeneous boundary value problem for composite region consisting of k-layers and it has been applied to solve the problems of heat conduction and elastic vibrations in composite plates, cylinders or spheres.

In the present paper we extend the results established in [2] to the case of two variables by considering Sturn-Liouville problem for the prolate spheroidal geometry defined by

$$-1\leq\eta\leq 1,\ \ \xi_i\leq\xi\leq\xi_{i+1},\ \ i=1,\ 2,\ \cdots\cdots,\ l,$$

and thereby we have generalized the results given in [1]. The applications of the transform to the physical problems will be the subject matter of subsequent work.

2. Preliminary results

As regards the spheroidal wave functions the notations used are as given in Flammer [5].

i) Spheroidal Coordinates

The prolate spheroidal coordinates are related to the rectangular coordinates by the transformation [5, p.6]

$$x=\frac{d}{2}[(1-\eta^2)(\xi^2-1)]^{1/2}\cos\phi,$$

$$y=\frac{d}{2}[(1-\eta^2)(\xi^2-1)]^{1/2}\sin\phi,$$

$$z=\frac{d}{2}\eta\xi, \tag{2.1}$$

with $-1\leq\eta\leq1$, $1\leq\xi<\infty$, $0\leq\phi\leq2\pi$, where d is the interfocal distance. (2.2)

The oblate spheroidal coordinates are related to the rectangular coordinates by the transformation [5, p. 6]

$$x=\frac{d}{2}[1-\eta^2)(\xi^2+1)]^{1/2}\cos\phi,$$

$$y=\frac{d}{2}[(1-\eta^2)(\xi^2+1)]^{1/2}\sin\phi,$$

$$z=\frac{d}{2}\eta\xi \tag{2.3}$$

with either

$$-1\leq\eta\leq1,\ 0\leq\xi<\infty,\ 0\leq\phi\leq2\pi,\ \text{or}\ 0\leq\eta\leq1,\ -\infty<\xi<\infty,\ 0\leq\phi\leq2\pi. \tag{2.4}$$

ii) The Spheroidal Differential Equations

To express the scalar wave equation $(\nabla^2+k^2)=0$ in spheroidal coordinates, we need the metrical coefficients h_η, h_ξ, h_ϕ which are defined by [5, p. 10].

$$dx^2+dy^2+dz^2=h_\eta^2 d\eta^2+h_\xi^2 d\xi+h_\phi^2 d\phi^2 \tag{2.5}$$

These scale factors are respectively,

$$h_\eta=\frac{d}{2}\left[\frac{\xi^2-\eta^2}{1-\eta^2}\right]^{1/2},$$

$$h_\xi=\frac{d}{2}\left[\frac{\xi^2-\eta^2}{\xi^2-1}\right]^{1/2},$$

$$h_\phi=\frac{d}{2}[(1-\eta^2)(\xi^2-1)]^{1/2}, \tag{2.6}$$

and

$$h_\eta=\frac{d}{2}\left[\frac{\eta^2+\xi^2}{1-\eta^2}\right]^{1/2},$$

$$h_\xi=\frac{d}{2}\left[\frac{\eta^2+\xi^2}{\xi^2+1}\right]^{1/2}, \tag{2.7}$$

$$h_\phi=\frac{d}{2}[(1-\eta^2)(\xi^2+1)]^{1/2},$$

in the prolate and oblate systems. With the use of the expression for the Laplacian ∇^2 in orthogonal curvilinear coordinates, we obtain the equations

$$\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}+c^3(\xi^2-\eta^2)\right]\phi=0, \tag{2.8}$$

and

$$\left[\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2+1)\frac{\partial}{\partial\xi}+\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2}{\partial\phi^2}+c^2(\xi^2+\eta^2)\right]\phi=0, \tag{2.9}$$

in the prolate and oblate cases respectively. In these equations we have set

$$c=\frac{1}{2}kd. \tag{2.10}$$

It is important to note that by transformations

$$\xi\to\pm i\xi,\ c=\pm ic, \tag{2.11}$$

we obtain oblate system from prolate system and vice-versa.

By the usual procedure of separation of variables, solutions of 2.8 and 2.9 may be obtained in the Lame products,

$$\phi_{mn}=S_{mn}(c,\eta)\,R_{mn}(c,\eta)\,{\cos \atop \sin}\,m\phi, \tag{2.12}$$

and

$$\phi_{mn}=S_{mn}(-ic,\eta)R_{mn}(-ic,\ i\eta)\,{\cos \atop \sin}\,m\phi, \tag{2.13}$$

respectively.

The four solutions $S_{mn}(c,\eta)$, $R_{mn}(c,\xi)$, $S_{mn}(-ic,\eta)$, and $R_{mn}(-ic,i\xi)$ satisfy the ordinary differential equations

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]+\left[\lambda_{mn}-c^2\eta^2-\frac{m^2}{1-\eta^2}\right]S_{mn}(c,\eta)=0 \tag{2.14}$$

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]-\left[\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right]R_{mn}(c,\xi)=0 \tag{2.15}$$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(-ic,\eta)\right]+\left[\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right]S_{mn}(-ic,\eta)=0 \tag{2.16}$$

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi}R_{mn}(-ic,i)\right]-\left[\lambda_{mn}-c^2\xi^2-\frac{m^2}{\xi^2+1}\right]R_{mn}(-ic,i\xi)=0 \tag{2.17}$$

The separation constants λ_{mn} and m are the same in the first of these equations, and like-wise in the second pair.

iii) Prolate spheroidal angle function of first kind is given by [5, p. 16]

$$S^{(1)}_{mn}(c,\eta)=\sum_{r=0,1}^{\infty}d^{mn}_r(c)\,P^m_{m+r}(\eta) \tag{2.18}$$

Here and in the sequel, the prime over the summation sign indicates that the summation is over only even values of r when $n-m$ is even, and over only odd values of r when $n-m$ is odd.

iv) Prolate spheroidal angle function of second kind is given by [5, p. 26]

$$S_{mn}^{(2)}(c, z)=\sum_{r=-\infty}^{\infty} d_r^{mn}(c)Q_{m+r}^{m}(z) \tag{2.19}$$

v) Prolate spheroidal radial function of first kind is given by [5, p. 31]

$$R_{mn}^{(1)}(c, \xi)=\frac{1}{\sum\limits_{r=0,1}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!}}\left(\frac{\xi^2-1}{2}\right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c)\frac{(2m+r)!}{r!} j_{m+r}(c\xi) \tag{2.20}$$

where $j_{m+r}(c\xi)$ is the spherical Bessel function of first kind

vi) Prolate spheroidal radial function of second kind is given by [5, p. 32]

$$R_{mn}^{(2)}(c, \xi)=\frac{1}{\sum\limits_{r=0,1}^{\infty} d_r^{mn}(c)\frac{(2m+r)!}{r!}}\left(\frac{\xi^2-1}{2}\right)^{m/2} \times \sum_{r=0,1}^{\infty} i^{r+m-n} d_r^{mn}(c)\frac{(2m+r)}{r} n^{m+r}(c\xi), \tag{2.21}$$

where $n_{m+r}(c\xi)$ is the spherical Bessel function of second kind.

vii) Recursion formula for $d_r^{mn}(c)$ is given by [5, p. 17]

$$\frac{(2m+r+2)(2m+r+1)}{(2m+2r+3)(2m+2r+5)}c^2 d_{r+2}^{mn}(c) + \left[(m+r)(m+r+1)-\lambda_{mn}(c)+\frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}c^2\right]d_r^{mn}(c) + \frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}d_{r-2}^{mn}(c)=0, \quad (r\geq 0). \tag{2.22}$$

viii) Normalization constant is given by [5, p. 22]

$$\int_{-1}^{1} S_{mn}(c, \eta)S_{mn'}(c, \eta)d\eta=\delta_{nn'}N_{mn},$$

where

$$N_{mn}=2\sum_{r=0,1}^{\infty}\frac{(r+2m)!\,(d_r^{mn})^2}{(2r+2m+1)\,r!} \tag{2.23}$$

3. Sturm-Liouville problem for composite prolate spheroidal region

Let us consider a system of equations

$$\left[\alpha_i\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}-\frac{m^2(\xi^2-\eta^2)}{(\xi^2-1)(1-\eta^2)}\right\}+\beta_i c_n^2(\xi^2-\eta^2)\right]X_{m,n,i}(\eta,\xi)=0,$$

$$-1\leq\eta\leq 1,\ \xi_i\leq\xi\leq\xi_{i+1},\ i=1,\ 2,\ \cdots\ l, \tag{3.1}$$

subject to the boundary and interfacial conditions

$$\left[-\alpha_1\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)+h_0X_{m,n,i}(\eta,\xi)\right]_{\xi=\xi_1}=0,\ \ h_0\geq 0,$$

$$\left[\alpha_l\frac{\partial}{\partial\xi}X_{m,n,l}(\eta,\xi)+h_l X_{m,n,l}(\eta,\xi)\right]_{\xi=\xi_{l+1}}=0,\ \ h_l\geq 0,$$

$$\alpha_i\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big|_{\xi=\xi_{i+1}}=\alpha_{i+1}\frac{\partial}{\partial\xi}X_{m,n,i+1}(\eta,\xi)\Big|_{\eta=\xi_{i+1}}$$

$$=\frac{1}{R_i}[X_{m,n,i+1}(\eta,\xi)-X_{m,n,i}(\eta,\xi)]_{\xi=\xi_{i+1}},\ \ i=1,2\ \cdots\ (l-1), \tag{3.2}$$

where

c_n^2 -Eigenvalue of the problem

α_i, β_i -Characteristics of i^{th} layer

R_i -Characteristic of i^{th} interface

h_0 -Surface coefficient at $\xi=\xi_1$

h_l -Surface coefficient at $\xi=\xi_{l+1}$.

Equations (3.1) and (3.2) constitute the mathematical formulation of the Sturm-Liouville problem for composite prolate spheroidal geometry defined by $(-1\leq\eta\leq 1,\ \xi_i\leq\xi\leq\xi_{i+1},\ i=1,\ 2,\ \cdots,\ l)$.

4. Solution of the problem

Since the separated differential equations in variables η and ξ are each of second order, the solution of (3.1) will involve prolate spheroidal angle functions of first and second kind and prolate spheroidal radial functions of first and second kind given in (2.18), (2.19), (2.20) and (2.21) respectively.

Hence the general solution of (3.1) is

$$X_{m,n,i}(\eta,\xi)=\left[A_{in}^1 R_{mn}^{(1)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \xi\right)+A_{in}^2 R_{mn}^{(2)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\xi\right)\right]$$

$$\times\left[A_{in}^{3}S_{mn}^{(1)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \eta\right)+A_{in}^{4}S_{mn}^{(2)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \eta\right)\right] \tag{4.1}$$

$$\xi_i\leq\xi\leq\xi_{i+1},\ i=1,\ 2,\ \cdots,\ l,\ -1\leq\eta\leq1,\ n=1,\ 2,\ \cdots,$$

where $A_{in}^{j}(j=1,\ 2,\ 3$ and $4)$ are arbitrary constants.

The function $S_{mn}^{(2)}(c_n,\ \eta)$ is singular at $\eta=\pm1$ and hence A_{in}^{4} must be zero for the finite form of the solution (4.1).

So the general solution (4.1) becomes

$$X_{m,n,i}(\eta,\xi)=\left[M_{i,n}R_{mn}^{(1)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \xi\right)+N_{i,n}R_{mn}^{(2)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \xi\right)\right]S_{mn}^{(1)}\left(\sqrt{\frac{\alpha_i}{\beta_i}}c_n,\ \eta\right) \tag{4.2}$$

$$-1\leq\eta\leq1,\ \xi_i\leq\xi\leq\xi_{i+1},\ i=1,\ 2\ \cdots\ l,\ n=1,\ 2,\ \cdots,$$

where $M_{i,n}$ and $N_{i,n}$ are arbitrary constants.

Substituing the solution (4.2) in the equations (3.2), we get a system of 21 simultaneous equations and from these we can calculate $M_{i,n}$ and $N_{i,n}$. Also from the system of 21 simultaneous equations on eliminating $M_{i,n}$ and $N_{i,n}$, we get the frequency equation. After substituting the values of $N_{i,n}$ and $N_{i,n}$, we get the required solution of the Sturm-Liouville problem (3.1) subject to the boundary and interfacial conditions (3.2).

5. Orthogonality of eigenfunctions $X_{m,n,i}(\eta,\xi)$

Let $X_{m,n,i}(\eta,\xi)$ and $X_{m,j,i}(\eta,\xi)$ be the two solutions of (3.1), then

$$\frac{\alpha_i}{(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}-\frac{m^2}{(\xi^2-1)(1-\eta^2)}\right\}X_{m,n,i}(\eta,\xi)$$
$$+\beta_i\,c_n^2\,X_{m,n,i}(\eta,\xi)=0, \tag{5.1}$$

and

$$\frac{\alpha^i}{(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}-\frac{m^2}{(\xi^2-1)(1-\eta^2)}\right\}X_{m,n,i}(\eta,\xi)$$
$$+\beta_i\,c_j^2\,X_{m,j,i}(\eta,\xi)=0, \tag{5.2}$$
$$-1\leq\eta\leq1,\ \xi_i\leq\xi\leq\xi_{i+1},\ i=1,\ 2,\ \cdots,\ l.$$

We multiply (5.1) by $(\xi^2-\eta^2)X_{m,j,i}(\eta,\ \xi)$, (5.2) by $(\xi^2-\eta^2)X_{m,n,i}(\eta,\ \xi)$, subtract and integrate with respct to η and ξ in the above intervals to obtain

$$\sum_{i=1}^{l}\left\{\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}\left[X_{m,j,i}(\eta,\xi)\frac{\partial}{\partial\eta}\left\{(1-\eta^2)\frac{\partial}{\partial\eta}X_{m,n,i}(\eta,\xi)\right\}\right.\right.$$

$$-X_{m,n,i}(\eta,\xi)\frac{\partial}{\partial\eta}\Big\{(1-\eta^2)\frac{\partial}{\partial\eta}X_{m,j,i}(\eta,\xi)\Big\}\Big]d\xi d\eta\Big\}$$

$$+\sum_{i=1}^{l}\Big\{\alpha_i\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}\Big[X_{m,j,i}(\eta,\xi)\frac{\partial}{\partial\xi}\Big\{(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big\}$$

$$-X_{m,n,i}(\eta,\xi)\frac{\partial}{\partial\xi}\Big\{(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,j,i}(\eta,\xi)\Big\}\Big]d\xi d\eta\Big\}$$

$$=\sum_{i=1}^{l}\beta_i(c_j^2-c_n^2)\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}(\xi^2-\eta^2)X_{m,n,i}(\eta,\xi)X_{m,j,i}(\eta,\xi)d\xi d\eta. \tag{5.3}$$

Let I_1 and I_2 denote the first and second group of terms in the left hand side of (5.3).

Integrating by parts with respect to η, the value of I_1 can be easily proved to be zero.

In I_2 changing the order of integration and on integration by parts with respect to ξ, we obtain

$$I_2=\sum_{i=1}^{l}\int_{-1}^{1}\alpha_i\Big[X_{m,j,i}(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)$$

$$-X_{m,n,i}(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,j,i}(\eta,\xi)\Big]_{\xi=\xi_i}^{\xi=\xi_{i+1}}d\eta.$$

On changing the order of integration and summation, we obtain

$$I_2=\int_{-1}^{1}\Big[\alpha_l X_{m,j,l}(\eta,\xi_{l+1})(\xi^2_{l+1}-1)\Big\{\frac{\partial}{\partial\xi}X_{m,n,l}(\eta,\xi)\Big\}_{\xi=\xi_{l+1}}$$

$$-\alpha_l X_{m,n,l}(\eta,\xi_{l+1})\Big\{(\xi_{l+1}^2-1)\frac{\partial}{\partial\xi}X_{m,j,l}(\eta,\xi)\Big\}_{\xi=\xi_{l+1}}\Big]d\eta$$

$$+\int_{-1}^{1}\sum_{i=1}^{l-1}\Big[\Big\{\alpha_i X_{m,j,i}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\alpha_i X_{m,n,i}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,j,i}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\Big\{\alpha_{i+1}X_{m,j,i+1}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,n,i+1}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\alpha_{i+1}X_{m,n,i+1}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,j,i+1}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}\Big\}\Big]d\eta$$

$$-\int_{-1}^{1}\alpha_1 X_{m,j,1}(\eta,\xi_1)(\xi_1^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,n,1}(\eta,\xi)\Big]_{\xi=\xi_1}$$

$$-\alpha_1 X_{m,n,1}(\eta,\xi_1)(\xi_1^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,j,1}(\eta,\xi)\Big]_{\xi=\xi_1}d\eta.$$

Using the boundary and interfacial conditions for the eigenfunctions $X_{m,n,i}(\eta,\xi)$ and $X_{m,j,i}(\eta,\xi)$, we have

$$I_2=\int_{-1}^{1}\Big[(\xi^2_{l+1}-1)\Big\{\{X_{m,j,l}(\eta,\xi_{l+1})\Big[[-h_l X_{m,n,l}(\eta,\xi_{l+1}]\Big]\Big\}$$

$$-(\xi^2_{l+1}-1)\Big\{\{X_{m,n,l}(\eta,\xi_{l+1})\Big[-h_l X_{m,j,l}(\eta,\xi_{l+1})\Big]\Big\}\Big]d\eta$$

$$+\int_{-1}^{1}\sum_{i=1}^{l-1}\Big[(\xi^2_{i+1}-1)\Big[\Big\{X_{m,j,i}(\eta,\xi_{i+1})\frac{1}{R_i}\Big[X_{m,n,i+1}(\eta,\xi_{i+1})$$

$$-X_{m,n,i}(\eta,\xi_{i+1})\Big]-X_{m,n,i}(\eta,\xi_{i+1})\frac{1}{R_i}\Big[X_{m,j,i+1}(\eta,\xi_{i+1})-X_{m,j,i}(\eta,\xi_{i+1})\Big]\Big\}$$

$$-\Big\{X_{m,j,i+1}(\eta,\xi_{i+1})\frac{1}{R_i}\Big[X_{m,n,i+1}(\eta,\xi_{i+1})-X_{m,n,i}(\eta,\xi_{i+1})\Big]$$

$$-X_{m,n,i+1}(\eta,\xi_{i+1})\frac{1}{R_i}\Big[X_{m,j,i+1}(\eta,\xi_{i+1})-X_{m,j,i}(\eta,\xi_{i+1})\Big]\Big\}\Big]\Big]d\eta$$

$$-\int_{-1}^{1}\Big[(\xi^2_1-1)\Big\{X_{m,j,1}(\eta,\xi_1)h_0X_{m,n,1}(\eta,\xi_1)-X_{m,n,1}(\eta,\xi_1)h_0X_{m,j,1}(\eta,\xi_1)\Big\}\Big]d\eta$$

$$=0.$$

Hence from (5.3) an orthogonality relation of the eigenfunctions $X_{m,n,i}(\eta,\xi)$ can be expressed as

$$A(n)\Big[\sum_{i=1}^{l}\beta_i\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}(\xi^2-\eta^2)X_{m,n,i}(\eta,\xi)X_{m,j,i}(\eta,\xi)d\xi d\eta\Big]=\delta_{nj} \qquad (5.4)$$

where $\sqrt{A(n)}$ is the normalising constant and δ_{nj} is the Kroneker delta which takes the value zero if $n\neq j$ and unity if $n=j$.

6. Definition and the inversion of the transform

We define the integral transform by the system

$$u_i^*(c_n)=\beta_i\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}(\xi^2-\eta^2)X_{m,n,i}(\eta,\xi)u_i(\eta,\xi)d\xi d\eta \qquad (6.1)$$

$$i=1,2,\dots,l,$$

where $u_i^*(c_n)$ is the integral transform of $u_i(\eta,\xi)$ with respect to the kernel $X_{m,n,i}(\eta,\xi)$ and weight function $(\xi^2-\eta^2)$.

In view of the completeness of the eigenfunction expansion and the orthogonal property (5.4), the inverse transform of (6.1) is readily obtained, provided that $u_i(\eta, \xi)$ $(i=1, 2 \dots l)$ are continuous and have piecewise continuous first and second derivatives in $(-1 \le \eta \le 1,\ \xi_i \le \xi \le \xi_{i+1},\ i=1, 2, \dots, l)$ and satisfy boundary and interfacial conditions of the eigenvalue problem [6-p. 293], as

$$u_i(\eta, \xi) = \sum_n A(n) X_{m,n,i}(\eta, \xi) \sum_{i=1}^{l} u_i^*(c_n) \tag{6.2}$$

$$-1 \le \eta \le 1,\ \xi_1 \le \xi \le \xi_{i+1},\ i=1, 2, \dots l,\ n=1.2, 3, \dots,$$

where the coefficients $A(n)$ are obtained from the relation (5.4).

7. Property of the transform

Let us consider the effect of the integral transform defined in (6.1) on the group of terms

$$\frac{\alpha_i}{\beta_i(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial}{\partial \eta}+\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial}{\partial \xi}-\frac{m^2}{(\xi^2-1)(1-\eta^2)}\right\}u_i(\eta, \xi) \tag{7.1}$$

$$-1 \le \eta \le 1,\ \xi_i \le \xi \le \xi_{i+1},\ i=1, 2, 3, \dots, l.$$

We have

$$I = \sum_{i=1}^{l} \int_{\xi_i}^{\xi_{i+1}} \int_{-1}^{1} \alpha_i X_{m,n,i}(\eta, \xi)\left[\frac{\partial}{\partial \eta}\left\{(1-\eta^2)\frac{\partial}{\partial \eta}u_i(\eta, \xi)\right\}+\frac{\partial}{\partial \xi}\left\{(\xi^2-1)\frac{\partial}{\partial \xi}u_i(\eta, \xi)\right]d\xi d\eta$$

$$= \sum_{i=1}^{l} \int_{\xi_i}^{\xi_{i+1}} \int_{-1}^{1} \int \alpha_i X_{m,n,i}(\eta, \xi)\frac{\partial}{\partial \eta}\left\{(1-\eta^2)\frac{\partial}{\partial \eta}u_i(\eta, \xi)d\xi d\eta\right.$$

$$+\sum_{i=1}^{l} \int_{\xi_i}^{\xi_{i+1}} \int_{-1}^{1} \alpha_i X_{m,n,i}(\eta, \xi)\frac{\partial}{\partial \xi}\left\{(\xi^2-1)\frac{\partial}{\partial \xi}u_i(\eta, \xi)\right\}d\xi d\eta. \tag{7.2}$$

Let I_1 and I_2 denote the first and second group of terms respectively in the right hand side of (7.2).

Integrating by parts twice with respect to η, we obtain

$$I_1 = \sum_{i=1}^{l} \int_{\xi_i}^{\xi_{i+1}} \int_{-1}^{1} \alpha_i u_i(\eta, \xi)\frac{\partial}{\partial \eta}\left\{(1-\eta^2)\frac{\partial}{\partial \eta}X_{m,n,i}(\eta, \xi)\right\}d\xi d\eta \tag{7.3}$$

Changing the order of integration in I_2 and integrating by parts with respect to ξ twice, we have

$$I_2 = \sum_{i=1}^{l}\left\{\int_{-1}^{1}\left[\alpha_i X_{m,n,i}(\eta, \xi)(\xi^2-1)\frac{\partial}{\partial \xi}u_i(\eta, \xi)-\alpha_i u_i(\eta, \xi)(\xi^2-1)\frac{\partial}{\partial \xi}X_{m,n,i}(\eta, \xi)\right]_{\xi_i}^{\xi_{i+1}} d\eta\right.$$

$$+\int_{-1}^{1}\int_{\xi_i}^{\xi_{i+1}}\alpha_i u_i(\eta,\xi)\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big\}d\eta d\xi.$$

Changing the order of integration in the last integral above, we get

$$I_2=\sum_{i=1}^{l}\Big\{\int_{-1}^{1}\Big[\alpha_i X_{m,n,i}(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}u_i(\eta,\xi)$$

$$-\alpha_i u_i(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big]_{\xi_i}^{\xi_{i+1}}d\eta$$

$$+\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}\alpha_i u_i(\eta,\xi)\frac{\partial}{\partial\xi}\Big\{(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big\}d\xi d\eta\Big\} \tag{7.4}$$

Substituting the values of I_1 (7.3) and I_2 (7.4) in (7.2) and using the fact that $X_{m,n,i}(\eta,\xi)$ is the solution of (3.1), we obtain

$$\sum_{i=1}^{l}\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}\alpha_i X_{m,n,i}(\eta,\xi)\Big[\frac{\partial}{\partial\eta}\Big\{(1-\eta^2)\frac{\partial}{\partial\eta}u_i(\eta,\xi)\Big\}$$

$$+\frac{\partial}{\partial\xi}\Big\{(\xi^2-1)\frac{\partial}{\partial\xi}u_i(\eta,\xi)\Big\}-\frac{m^2u_i(\eta,\xi)}{(\xi^2-1)(1-\eta^2)}\Big]d\xi d\eta$$

$$=\sum_{i=1}^{l}\int_{-1}^{1}\Big[\alpha_i X_{m,n,i}(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}u_i(\eta,\xi)$$

$$-\alpha_i u_i(\eta,\xi)(\xi^2-1)\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big]_{\xi_i}^{\xi_{i+1}}d\eta-c_n^2\sum_{i=1}^{1}u_i^*(c_n).$$

Changing the order of sumation and integration in the right hand side of above, we have

$$=\int_{-1}^{1}\Big\{\alpha_l X_{m,n,l}(\eta,\xi_{l+1})(\xi_{l+1}^2-1)\Big[\frac{\partial}{\partial\xi}u_l(\eta,\xi)-\frac{\frac{\partial}{\partial\xi}X_{m,n,l}(\eta,\xi)}{X_{m,n,l}(\eta,\xi)}\Big]_{\xi=\xi_{l+1}}\Big\}d\eta$$

$$+\int_{-1}^{1}\Big\{\sum_{i=1}^{l-1}\Big[\Big\{\alpha_i X_{m,n,i}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}u_i(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\alpha_i u_i(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,n,i}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\Big\{\alpha_{i+1}X_{m,n,i+1}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}u_{i+1}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}$$

$$-\alpha_{i+1}u_{i+1})(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big[\frac{\partial}{\partial\xi}X_{m,n,i+1}(\eta,\xi)\Big]_{\xi=\xi_{i+1}}\Big\}\Big]\Big\}d\eta$$

$$-\int_{-1}^{1}\Big\{\alpha_1 X_{m,n,1}(\eta,\xi_1)(\xi_1^2-1)\Big[\frac{\partial}{\partial\xi}u_1(\eta,\xi)-\frac{\frac{\partial}{\partial\xi}X_{m,n,1}(\eta,\xi)}{X_{m,n,1}(\eta,\xi)}\Big]_{\xi=\xi_1}\Big\}d\eta$$

$$-c_n^2\sum_{i=1}^{l}u_i^*(c^n). \tag{7.5}$$

Using boundary, interfacial conditions (3.2) in (7.5) and rearranging the terms, we get the required property of the transform as

$$\sum_{i=1}^{l}\int_{\xi_i}^{\xi_{i+1}}\int_{-1}^{1}\beta_i(\xi^2-\eta^2)X_{m,n,i}(\eta,\xi)\Big[\frac{\alpha_i}{\beta_i(\xi^2-\eta^2)}\Big\{\frac{\partial}{\partial\eta}(1-\eta^2)\frac{\partial}{\partial\eta}$$

$$+\frac{\partial}{\partial\xi}(\xi^2-1)\frac{\partial}{\partial\xi}-\frac{m^2}{(\xi^2-1)(1-\eta^2)}\Big\}u_i(\eta,\xi)\Big]d\xi d\eta$$

$$=\int_{-1}^{1}X_{m,n,l}(\eta,\xi_{l+1})(\xi_{l+1}^2-1)\Big[\alpha_l\frac{\partial}{\partial\xi}u_i(\eta,\xi)+h_lu_i(\eta,\xi)\Big]_{\xi=\xi_{l+1}}d\eta$$

$$+\int_{-1}^{1}\Big\{\sum_{i=1}^{l-1}\Big[X_{m,n,i}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big\{\alpha_i\frac{\partial}{\partial\xi}u_i(\eta,\xi)-\frac{1}{R_i}\Big[u_{i+1}(\eta,\xi)-u_i(\eta,\xi)\Big]\Big\}_{\xi=\xi_{i+1}}$$

$$-X_{m,n,i+1}(\eta,\xi_{i+1})(\xi_{i+1}^2-1)\Big\{\alpha_{i+1}\frac{\partial}{\partial\xi}u_{i+1}(\eta,\xi)-\frac{1}{R_i}\Big[u_{i+1}(\eta,\xi)$$

$$-u_i(\eta,\xi)\Big]\Big\}_{\xi=\xi_{i+1}}\Big]\Big\}d\eta-\int_{-1}^{1}\Big\{X_{m,n,1}(\eta,\xi_1)(\xi_1^2-1)\Big[\alpha_1\frac{\partial}{\partial\xi}u_1(\eta,\xi)$$

$$-h_0u_1(\eta,\xi)\Big]_{\xi=\xi_1}\Big\}d\eta-c_n^2\sum_{i=1}^{l}u_i^*(c_n) \tag{7.6}$$

Hence (7.6) is the fundamental property of the integral transform defined in (6.1), which removes the group of terms quoted in (7.1).

8. Discussion

The integral transform defined in (6.1) is nothing but the generalization of the transforms defined in [1] and the results in [1] can be deduced from (6.1) by specializing the coefficients and parameters involved their-in.

Further it is observed that the range of integration for η can be made (0, 1) by adding one more condition

$$u_i(\eta,\xi)\Big|_{\eta=0}=0 \quad\text{or}\quad \frac{\partial}{\partial\eta}u_i(\eta,\xi)\Big|_{\eta=0},$$

in which case the transform defined in (6.1) will be called the odd or even transform respectively.

We can also consider the transfrom for the region ($\eta_j\leq\eta\leq\eta_{j+1}$, $\xi_i\leq\xi\leq\xi_{i+1}$, $j=1,2,\ldots,k$, $i=1,2,\ldots,l$) following the procedure given in [2].

The results presented here can be extended to oblate systems by using transformations (2.11).

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REFERENCES

[1] Gupta, R.K., *A finite transform involving spheroidal wave functions and its application*, Procd. of National Inst. of India, Vol·34, A, No.6, pp.289—300, 1968.

[2] Wankhede, P.C. and Bhonsle, B.R., *Sturm-Liouville transform for composite region*, To be published.

[3] Wankhede, P.C. and Bhonsle, B.R., *A transient heat conduction in composite plates, cylinders or spheres*, To be published.

[4] Wankhede, P.C. and Bhonsle, B.R., *Elastic Vibrations in Composite Cylinders or Spheres*, To be published.

[5] Flammer, C., *Spheroidal Wave Functions*, Stanford Uni. Press, 1957.

[6] Courant, R. and Hilbert, D., *Methods of Mathematical Physics*, vol.1, Interscience, 1953.