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ON SET-CONNECTED MAPPINGS

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1. Introduction

In [2] Jin Ho Kwak introduced a new class of mappings, called set-connected mappings, which contains the class of continuous mappings. He investigated several properties concerning such mappings, and among them, gave some sufficient conditions for such mappings to be continuous. The purpose of this note is to show that every weakly continuous surjection is set-connected but not conversely, and to give a sufficient condition for set-connected mappings to be continuous.

Throughout this note, X and Y always represent topological spaces and $f: X \rightarrow Y$ denotes a mapping (not necessarily continuous) f of a topological space X into a topological space Y. Let A be a subset of a topological space. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively.

2. Set-connected mappings

DEFINITION 1. A space X is said to be *connected between A and B* [1, p. 142] if there exists no closed-open set F of X such that $A \subset F$ and $F \cap B = \phi$.

DEFINITION 2. A mapping $f: X \rightarrow Y$ is said to be *set-connected* [2] provided that: if X is connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$ with respect to the relative topology.

The following lemma, due to Jin Ho Kwak [2], is very useful.

LEMMA 1 (Kwak, [2]). *A mapping $f: X \rightarrow Y$ is set-connected if and only if for any closed-open set F of $f(X)$, $f^{-1}(F)$ is closed-open in X.*

THEOREM 1. *Let $f: X \rightarrow Y$ be a mapping and $g: X \rightarrow X \times Y$ the graph mapping of f defined by $g(x) = (x, f(x))$ for each point $x \in X$. If g is set-connected, then f is set-connected.*

PROOF. Let F be any closed-open set of the subspace $f(X) \subset Y$. Then $X \times F$ is a closed-open set of the product space $X \times f(X)$. Now, the product space $X \times f(X)$

coincides with the subspace $X\times f(X)\subset X\times Y$. Thus $X\times F$ is closed-open in the subspace $X\times f(X)$. Since $g(X)$ is a subset of $X\times f(X)$. $(X\times F)\cap g(X)$ is a closed-open set of the subspace $g(X)\subset X\times Y$. Since $g:X\to X\times Y$ is a set-connected mapping, by Lemma 1, $g^{-1}((X\times F)\cap g(X))$ is a closed-open set of X. It follows from $g^{-1}((X\times F)\cap g(X))=g^{-1}(X\times F)=f^{-1}(F)$ that $f^{-1}(F)$ is closed-open in X. Therefore, again by Lemma 1, we observe that f is a set-connected mapping.

REMARK 1. The converse to Theorem 1 is not true, as the following example due to Kwak [2] shows.

EXAMPLE 1. Let X be the real numbers with the usual topology and $f:X\to X$ be a mapping defined by $f(x)=x^2$ if $x\neq 0$ and $f(x)=1$ if $x=0$. Then f is set-connected [2, Example 1]. The graph mapping $g:X\to X\times Y$ is, however, not set-connected.

We shall give a sufficient condition for set-connected mappings to be continuous. For this purpose, we need the following definitions.

DEFINITION 3. A space X is said to be *extremally disconnected* [7, p. 106] if the closure of every open set in X is open.

DEFINITION 4. A space X is said to be *C-compact* [6] if every cover of any closed set in X by open sets of X has a finite subfamily whose union is dense in the closed set.

THEOREM 2. *Let Y be an extremally disconnected, C-compact and Hausdorff space. If a mapping $f:X\to Y$ is set-connected, then f is continuous.*

PROOF. Suppose that f is not continuous. Then there exists a closed set F of Y such that $f^{-1}(F)$ is not closed in X. Since $f^{-1}(F)$ is not closed in X, there exists a point $x\in \mathrm{Cl}(f^{-1}(F))-f^{-1}(F)$. Thus X is connected between x and $f^{-1}(F)$. Since f is set-connected, $f(X)$ is connected between $f(x)$ and $f(f^{-1}(F))$. By Theorem 1a and Theorem 1d of [1, p. 143], we obtain that Y is connected between $f(x)$ and F. Now, since X is Hausdorff, for each point $y\in F$ there exists an open neighborhood V_y of y in Y such that $f(x)\not\in \mathrm{Cl}(V_y)$. The family $\{V_y\mid y\in F\}$ is a cover of F by open sets of Y. Since F is closed in the C-compact space Y, there exist a finite number of points $y_1, y_2, \dots, y_n$ in F such that $F\subset \mathrm{Cl}(\cup V_{y_j}\mid 1\leqq j\leqq n\})$. Put $V=\mathrm{Cl}(\cup\{V_{y_j}\mid 1\leqq j\leqq n\})$. Then V is a closed-open set of Y because Y is extremally disconnected. Moreover, we obtain that $F\subset V$ and $f(x)$

$\notin V$. This contradicts that Y is connected between $f(x)$ and F. Therefore, f is continuous.

3. Weakly continuous mappings and set-connected mappings

DEFINITION 5. A mapping $f: X \to Y$ is said to be *weakly continuous* [3] if for each point $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U of X containing x such that $f(U) \subset \mathrm{Cl}(V)$.

REMARK 2. Every continuous mapping is weakly continuous, but the converse is not true [3].

LEMMA 2. (Levine, [3]) *A mapping $f: X \to Y$ is weakly continuous if and only if for each open set $V \subset Y$, $f^{-1}(V) \subset$ $\mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$.*

LEMMA 3. (Noiri, [4]). *If a mapping f: $X \to Y$ is weakly continuous, then* $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for each open set $V \subset Y$.*

THEOREM 3. *If a surjection $f: X \to Y$ is weakly continuous, then f is set-connected.*

PROOF. Let V be any closed-open set of Y. Since V is closed, We have $\mathrm{Cl}(V) = V$. Thus, by Lemma 2, we obtain $f^{-1}(V) \subset \mathrm{Int}\,(f^{-1}(V))$. This shows that $f^{-1}(V)$ is an open set of X. Moreover, by Lemma 3, we obtain $\mathrm{Cl}\,(f^{-1}(V)) \subset f^{-1}(V)$. This shows that $f^{-1}(V)$ is a closed set of X. Since f is surjective, by Lemma 1, we observe that f is a set-connected mapping.

REMARK 3. The converse to Theorem 3 is not true, as the following example due to N. Levine [3] shows.

EXAMPLE 2. Let X be the unit interval with the countable complement topology. Let Y be the unit interval with the usual topology and $f: X \to Y$ be the identity mapping. Then, since $f(X) = Y$ is connected, f is a set-connected mapping [2, Lemma 3]. However, f is not weakly continuous.

THEOREM 4. *Let Y be an extremally disconnected space. If a mapping $f: X \to Y$ is set-connected, then f is weakly continuous.*

PROOF. Let x be any point of X and V any open set of Y containing $f(x)$. Since Y is extremally disconnected, $\mathrm{Cl}(V)$ is closed-open in Y. Thus $\mathrm{Cl}(V) \cap f(X)$ is closed-open in the subspace $f(X)$. Put $f^{-1}(\mathrm{Cl}(V) \cap f(X)) = U$. Then, since f is set-connected, it follows from Lemma 1 that U is closed-open in X. Therefore, U is an open neighborhood of x in X such that $f(U)\subset \mathrm{Cl}(V)$. This implies that f is weakly continuous.

COROLLARY 1. *Let Y be an extremally disconnected space. A surjection $f: X\to Y$ is set-connected if and only if f is weakly continuous.*

PROOF. This is an immediate consequence of Theorem 3 and Theorem 4.

The author showed that if a mapping $f: X\to Y$ is weakly continuous and Y is Hausdorff, then the graph $G(f)$ of f is closed in $X\times Y$ [5, Theorem 10]. Therefore, the following corollary is an immediate consequence of Theorem 4.

COROLLARY 2 (Kwak, [3]). *If $f: X\to Y$ is a set-connected mapping and Y is extremally disconnected Hausdorff, then $G(f)$ is closed in $X\times Y$.*

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