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THE θ AND α-MONADS IN GENERAL TOPOLOGY

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1. Introduction

Since it was first introduced by Abraham Robinson [12], the point monad, $\mu(p)$ has proved to be a useful device for characterizing and studying numerous topological concepts. In this paper, we examine two new point monads; the θ-monad, α-monad, which are capable of similarly characterizing the various concepts associated with quasi-H-closed, nearly-compact (Singal [15]), semiregular, regular, almost-regular [14], T_2 and Urysohn spaces as well as θ, δ-cluster points θ, δ-convergence theory, θ-continuity, almost continuity [16], among others. Indeed, the θ-monad shall play an important role in our approach to H-closed, Urysohn spaces and unique cluster point theory. We shall also investigate the relations between the θ, δ-monads and the monad of Robinson showing how their interaction not only characterizes almost-regular, semiregular and regular spaces but allows use to immediately deduce interesting results about almost-open [16] weakly-open and open maps.

For a reasonably complete set of references relative to compact-like spaces, we refer the reader to Mathur's paper [10].

2. Basic definitions and preliminaries

Throughout this paper, we shall let L denote a suitable first order language with equality, $M=(U, \in, pr, ap)$ our standard model and $^*M=(^*U, {}^*\in, {}^*pr, {}^*ap)$ our enlargement of M [9]. For $A \subset X \in U$, we shall let $A^* = \{p \mid p \in {}^*U \text{ and } p\, {}^*\!\in A\}$. For other notation used in this paper and results in the nonstandard theory of filters, we refer the reader to [8], [9].

For a space (X, τ), we shall let τ_s denote the set of all regular-open subsets in X and (X, T_s), the space with the topology generated by the base τ_s. We shall refer to this topology as the semiregularization of τ. For $A \subset \mathscr{P}(X)$, $\langle A \rangle$ shall denote the filter generated by A. If $p \in X$, then $R(p) = \{G \mid G \in \tau_s \text{ and } p \in G\}$, $G(p) = \{G \mid G \in \tau \text{ and } p \in G\}$, $\overline{G(p)} = \{\mathrm{cl}\, G \mid G \in G(p)\}$, $N(p) = \{A \mid A \subset X \text{ and } A \text{ is a closed neighborhood of } p\}$. Clearly, $\langle \overline{G(p)} \rangle = \langle N(p) \rangle$ for each $p \in X$. If $\mathscr{N}(p)$ is the neighborhood filter for $p \in X$, then $<N(p)> \subset \mathscr{N}(p)$. Recall that a point $p \in X$ is a θ [resp. δ]-closure point of $A \subset X$ iff $G \cap A \neq \phi$ for each $G \in N(p)$ [resp. $G \in R(p)$]. If $\mathscr{F}$ is a filter on X, then $p \in X$ is a θ[resp. δ]-cluster point for $\mathscr{F}$ iff for every $G \in N(p)$[resp. $G \in R(p)$] and $F \in \mathscr{F}$, we have that $G \cap F \neq \phi$. We shall also say that the filter $\mathscr{F}$ is θ[resp. δ]-convergent to $p \in X$ if $\overline{G(p)}$ [resp. $R(p)$] $\subset \mathscr{F}$.

DEFINITION 2.1. For each $p \in X$, let the θ-monad $\mu_\theta(p) = \bigcap \{G^* \mid G \in \overline{G(p)}\}$ and the α-monad $\mu_\alpha(p) = \bigcap \{G^* \mid G \in R(p)\}$.

Clearly, for each $p \in X, \mu_\alpha(p) = \mu_s(p)$, where $\mu_s(p)$ is the monad of Robinson in (X, T_s). Notice that $\mu_\theta(p) = \text{Nuc} <N(p)>$ and $\mu(p) \subset \mu_\alpha(p) \subset \mu_\theta(p)$.

3. Quasi-H-closed and nearly-compact spaces

Since $\mu_\alpha(p) = \mu_s(p)$ for each $p \in X$, most of the nonstandard results for the α-monad in this paper are simple interpretations of the classical results found in [8], [9], [12] with respect to the semiregularization. For this reason, we shall often not present proofs for the results stated about the α-monads but rather leave them to the reader.

We shall now give the basic nonstandard characterizations for quasi-H-closed [11] (i.e. generalized absolutely closed [7]) and nearly-compact spaces.

THEOREM 3.1. *A space (X, τ) is quasi-H-closed [resp. nearly-compact] iff* $X^* = \bigcup \{\mu_\theta(p) \mid p \in X\}$ [*resp.* $X^* = \bigcup \{\mu_\alpha(p) \mid p \in X\}$].

PROOF. Recall that a space (X, τ) is quasi-H-closed iff every open cover Γ of X contains a finite subcover, say $\{G_1, \dots, G_n\}$, such that $X = \bigcup \{\text{cl } G_i \mid i = 1, \dots, n\}$. We assume that X is quasi-H-closed and that there exists some $q \in X^*$ such that $q \notin \mu_\theta(p)$ for any $p \in X$. Hence for each $p \in X$ there exists some $G_p \in G(p)$ such that $q {}^* \notin \text{cl } G_p$ from the definition of $\mu_\theta(p)$. Thus $D = \{G_p \mid p \in X,\ q {}^* \notin \text{cl } G_p$ and $G_p \in G(p)\}$ is an open cover o X. Now there exists a finite subset, say $\{G_1, \dots, G_n\}$, such that $X = \bigcup \{\text{cl } G_i \mid i = 1, \dots, n\}$. The sentence in L

$$\forall x [[x \in X] \to [x \in \text{cl } G_1] \vee \dots \vee [x \in \text{cl} G_n]]$$

holds in M; hence in ${}^*M$. Consequently, since $q\ {}^* \notin\ \text{cl} G_p$ for any $G_p \in D$ this would imply that $q\ {}^* \notin X$. A contradiction. Therefore, $q \in \mu_\theta(p)$ for some $p \in X$.

Conversely, assume that X is not quasi-H-closed. Let Γ be an open cover of X

such that no finite subset of Γ has closures which cover X. Let $K=\{\mathrm{cl}\ H \mid H\in\Gamma\}$. The well formed formula

$$\phi(x,y)=[X\in K]\wedge[y\in X]\wedge[\sim[y\in x]]$$

clearly determines a concurrent relation $R(\phi)$ on U. Thus there exists an $r(\phi)\in{}^*U$ such that $r(\phi)\in X^*$ and $r(\phi)\notin\mathrm{cl}\ H$ for each $\mathrm{cl}\ H\in K$. However, if $p\in X$, then $p\in H_1$ for some $H_1\in\Gamma$ which implies that $p\in\mathrm{cl}\ H_1\in K$. Evidently, $\mu_\theta(p)\subset(\mathrm{cl}\ H)^*$ from the definition of the θ-monad. Consequently, $r(\phi)\notin\mu_\theta(p)$ for any $p\in X$ and the conclusion follows.

COROLLARY 3.1. *A set* $A\subset X$ *is quasi-H-closed [resp. nearly-compact] relative to* X [11] *iff* $A^*\subset\bigcup\{\mu_\theta(p)\mid p\in A\}$ *[resp.* $A^*\subset\bigcup\{\mu_\alpha(p)\mid p\in A\}$*].*

We next relate quasi-H-closedness [resp. nearly-compactness] to θ [resp. δ] clustering of filters.

LEMMA 3.1. *Let* $\mathscr{F}$ *be a filter on a space* X. *Then* $p\in X$ *is a* θ *[resp.* δ*]-cluster point for* $\mathscr{F}$ *iff* $\mathrm{Nuc}\ \mathscr{F}\cap\mu_\theta(p)\neq\phi$ *[resp.* $\mathrm{Nuc}\ \mathscr{F}\cap\mu_\alpha(p)\neq\phi$*].*

PROOF. Since $\overline{G(p)}$ has the finite intersection property then $\overline{G(p)}$ generates a non-trivial filter $<\overline{G(p)}>$ on X. Thus $p\in X$ is a θ-cluster point of $\mathscr{F}$ iff the filter $<\mathscr{F}\cup<G(p)>>$ is non-trivial iff $\mathrm{Nuc}\mathscr{F}\cap\mathrm{Nuc}<\overline{G(p)}>=\mathrm{Nuc}\mathscr{F}\cap\mu_\theta(p)\neq\phi$.

LEMMA 3.2. *Let* $\mathscr{F}$ *be an filter on* X. *Then* $\mathscr{F}$ *is* θ *[resp.* δ*]-convergent to* $p\in X$ *iff* $\mathrm{Nuc}\ \mathscr{F}\subset\mu_\theta(p)$ *[resp.* $\mathrm{Nuc}\mathscr{F}\subset\mu_\alpha(p)$*].*

PROOF. Left to reader.

LEMMA 3.3. *Let* $p\in X$. *Then* p *is a cluster point of an open filter base* $\mathscr{F}$ *in* (X,τ) *iff* $\mathrm{Nuc}\mathscr{F}\cap\mu_\theta(p)\neq\phi$.

PROOF. The point $p\in X$ is a cluster point of $\mathscr{F}$ iff $p\in\mathrm{cl}\ G$ for each $G\in\mathscr{F}$. Now $p\in\mathrm{cl}\ G$ iff $G\cap H\neq\phi$ for each $H\in G(p)$. However $G\cap H\neq\phi$ iff $G\cap\mathrm{cl}H\neq\phi$. Hence p is a cluster point for $\mathscr{F}$ iff $G\cap\mathrm{cl}\ H\neq\phi$ for each $G\in\mathscr{F}$ and each $\mathrm{cl}\ H\in\overline{G(p)}$. All this implies that p is a cluster point for $\mathscr{F}$ iff the filter generated by $<\mathscr{F}>\cup<\overline{G(p)}>$ is non-trivial iff $\mathrm{Nuc}(<\mathscr{F}>\cup<\overline{G(p)}>)=\mathrm{Nuc}\mathscr{F}\cap\mathrm{Nuc}\ \overline{G(p)}=\mathrm{Nuc}\mathscr{F}\cap\mu_j(p)\neq\phi$.

THEOREM 3.2. *A set* $A\subset X$ *is quasi-H-closed [resp. nearly-compact] relative to* X *iff for every filter* $\mathscr{F}$ *on* A *there exists a* $p\in A$ *such that* $\mu_\theta(p)\cap\mathrm{Nuc}\mathscr{F}\neq\phi$ *[resp.* $\mu_\alpha(p)\cap\mathrm{Nuc}\mathscr{F}\neq\phi$*].*

PROOF. A straightforward application of the above lemmas.

4. Separation properties

In this section, we show how the monads $\mu(p)$, $\mu_\alpha(p)$, $\mu_\theta(p)$ interrelate in order to characterize regular, semiregular, almost-regular [14] spaces. First, we recall a few definitions. A space (X, τ) is semiregular iff $\tau = T_s$. A space (X, τ) is almost-regular iff for every regular-closed set $A \subset X$ and $p \in X - A$ there exists disjoint $G, H \in \tau$ such that $p \in G$ and $A \subset H$. A space X is Urysohn if for distinct $p, q \in X$ there exist neighboods N_p, N_q, of p, q, respectively, such that $\mathrm{cl} N_p \cap \mathrm{cl} N_q = \phi$.

THEOREM 4.1. *A space (X, τ) is Urysohn iff $\mu_\theta(p) \cap \mu_\theta(q) = \phi$ for each pair of distinct $p, q \in X$.*

PROOF. Assume that (X, τ) is Urysohn. Let $p, q \in X$ and $p \neq q$. Then there exists a pair of non-empty disjoint open sets, say G_1 and G_2, such that $p \in G_1$, $q \in G_2$ and $\mathrm{cl}\ G_1 \cap \mathrm{cl}\ G_2 = \phi$. From the definition on the "nucleus", it follows that $\mu_\theta(p) \subset (\mathrm{cl} G_1)^*$ and $\mu_\theta(q) \subset (\mathrm{cl}\ G_2)^*$. Thus $\mu_\theta(p) \cap \mu_\theta(q) = \phi$.

Conversely, if $\mu_\theta(p) \cap \mu_\theta(q) = \phi$ for distinct $p, q \in X$, then from the nonstandard theory of filters [8], [9] we have that there exists two infinitesimal *elements C, H which are *elements of $\overline{G(p)}$ and $\overline{G(q)}$, respectively. Therefore, we have that $p \in C^* \subset \mu_\theta(p)$, $q \in H^* \subset \mu_\theta(q)$ and $C^* \cap H^* = \phi$. The sentence in L

$$\exists x \exists y [[x \in \overline{G(p)}] \wedge [y \in \overline{G(q)}] \wedge [p \in x] \wedge [q \in y] \wedge [x \cap y = \phi]]$$

holds in $^*M$, hence in M. Interpreting this in M yields the desired result.

THEOREM 4.2. *Let (X, τ) be a topological space. Then*

(i) *X is regular iff $\mu_\theta(p) = \mu(p)$ for each $p \in X$.*

(ii) *X is semiregular iff $\mu_\alpha(p) = \mu(p)$ for each $p \in X$.*

(iii) *X is almost-regular iff $\mu_\theta(p) = \mu_\alpha(p)$ for each $p \in X$.*

(iv) *X is almost-regular iff for each $p \in X$ and $q \in X^*$ such that $q \notin \mu_\alpha(p)$ there exist disjoint regular-open G, H such that $p \in G$ and $q \in H^*$.*

PROOF. (i) One of the equivalent definitions for a regular space states that X is regular iff $N(p)$ is a base for $\mathscr{N}(p)$. This implies that X is regular iff Nuc $N(p) = \mathrm{Nuc}\mathscr{N}(p) = \mu_\theta(p) = \mu(p)$.

(ii) This follows immediately since $\tau = T_s$

(iii) The definition of almost-regular is clearly equivalent to the statement that for each $p\in X$ the set $N(p)$ is a base for $<R(p)>$. Consequently, X is almost-regular iff Nuc $N(p)=\mu_\theta(p)=\text{Nuc}<R(p)>=\mu_\alpha(p)$ for each $p\in X$.

(iv). This follows immediately by interpreting the classical result as found in [9] with respect to the semiregularization.

It is not surprising that the θ-monad which satisfactorily characterizes regular and Urysohn spaces should also characterize T_2 spaces.

THEOREM 4.3. *Let (X,τ) be a topological space. Then*

(i) *X is T_2 iff for each $p,q\in X$ whenever $q\in\mu_\theta(p)$, then $p=q$.*

(ii) *X is T_2 iff for distinct $p,q\in X$, we have that $\mu_\alpha(p)\cap\mu_\alpha(q)=\phi$.*

PROOF. (i) Assume that X is T_2 and that there exists $p,q\in X$ such that $q\in\mu_\theta(p)$ and $p\neq q$. Then there exist disjoint open sets, say G_1 and G_2, such that $p\in G_1$ and $q\in G_2$. Since cl $G_1\cap G_2=\phi$, $\mu_\theta(p)\subset(\text{cl } G_1)^*$, then $q\notin\mu_\theta(p)$. Evidently this contradiction yields the result.

Conversely, assume that for each $p,q\in X$, if $q\in\mu_\theta(p)$, then $p=q$. Now there exists some infinitesimal *element E $^*\in\overline{G(p)}$ such that $p\in E^*\subset\mu_\theta(p)$. If $q\in E^*$, then $q\in\mu_\theta(p)$and $q=p$ from the hypothesis. So if $p,q\in X$ and $p\neq q$, then $q\notin E^*$. Therefore, the sentence in L

$$\exists x[[x\in\overline{G(p)}]\wedge[p\in x]\wedge[\sim[q\in x]]$$

holds in $^*M$; hence in M. Let $C\in\overline{G(p)}$ such that $p\in C$ and $q\notin C$. From the definition of $\overline{G(p)}$ this implies that there exists a $G\in\tau$ such that $p\in G$ and $q\notin$cl G. Hence $q\in X$-cl $G\in\tau$ implies that X is T_2.

(ii) Since $\mu_\alpha(p)$ and $\mu_\alpha(q)$ are nuclei of τ_s filters then $\mu_\alpha(p)\cap\mu_\alpha(q)=\phi$ iff there exist some $R\in R(p)$ and $R_1\in R(q)$ such that $R\cap R_1=\phi$. However, $R=$int cl G and $R_1=$int cl G_1 for some $G,G_1\in\tau$. We know that $R\cap R_1=\phi$ iff $G\cap G_1=\phi$. Finally, $G\cap G_1=\phi$ iff $\mu(p)\cap\mu(q)=\phi$ iff X is T_2 [12].

We also relate semiregular spaces to an important clustering process.

THEOREM 4.4. *A space (X,τ) is semiregular iff whenever we have an open filter $\mathscr{F}$ and* Nuc$\mathscr{F}\subset\mu_\theta(p)$ *for some $p\in X$, then* Nuc$\mathscr{F}\subset\mu(p)$.

PROOF. Assume that X is semiregular, $\mathscr{F}$ is an open filter and Nuc $\mathscr{F}\subset\mu_\theta(p)$ for some $p\in X$. Let $\overline{R(p)}=\{\text{cl } G|G\in R(p)\}$. Then $\overline{R(p)}\subset\overline{G(p)}$ implies that Nuc $\mathscr{F}\subset$Nuc $\overline{R(p)}$. Hence for each $R\in\overline{R(p)}$ there exists some $F\in\mathscr{F}$ such that

$F \subset R$. However, $F \subset \mathrm{int\ cl}\ F$. Consequently, $F \subset \mathrm{int}\ R = G \in R(p)$. Hence, Nuc $\mathscr{F} \subset$ Nuc $R(p) = \mu_s(p) = \mu(p)$.

Conversely, assume that X is not semiregular. Then there exists some $p \in X$ such that $R(p)$ is not a base for $\mathscr{N}(p)$. Consequently, $\mu(p) \subset \mathrm{Nuc} R(p)$ and Nuc$R(p) - \mu(p) \neq \phi$. Let $q \in \mathrm{Nuc}\ R(p) - \mu(p)$. Then $\mathrm{H} = \{\mathrm{int\ cl}\ G \mid G \in \tau$ and $q \in G^*\}$ is an open filter base of regular-open sets. Clearly, Nuc$H \subset \mu(p)$. However, since $R(p) \subset H$ and $<\overline{G(p)}> \subset <R(p)>$, then Nuc$H \subset \mu_\theta(p)$.

COROLLARY 4.1. *The point* $p \in X$ *is semiregular iff whenever we have an open filter* $\mathscr{F}$ *and* Nuc $\mathscr{F} \subset \mu_\theta(p)$ *then* Nuc$\mathscr{F} \subset \mu(p)$.

REMARK. Theorem 4.2 (i) implies that in a non-discrete T_3 space $\mu_\theta(p)$ is an external subset in X^* for any nonisolated $p \in X$.

5. θ-continuity, almost-continuity

Basic to the study of θ, δ-clustering, quasi-H-closed and nearly-compact spaces are the concepts of θ-continuity, weak-θ-continuity [1], [6], strong-θ-continuity [5] and almost-continuity [16]. Also, the almost-open and weakly-open concepts are considered. We now present the nonstandard characterizations for theses variants of the continuous or open map $f: X \to Y$, where f^* will denote the unique nonstandard extension of f to X^*.

THEOREM 5.1. *Let* $f: (X, \tau) \to (Y, T)$. *Then*

(i) f *is* θ*-continuous at* $p \in X$ *iff* $f^*(\mu_\theta(p)) \subset \mu_\theta(f(p))$.

(ii) f *is weakly-*θ*-continuous* [1], [6] *at* $p \in X$ *iff* $f^*(\mu(p)) \subset \mu_\theta(f(p))$.

(iii) f *is strongly-continuous* [5] *at* $p \in X$ *iff* $f^*(\mu_\theta(p)) \subset \mu(f(p))$.

(iv) f *is almost continuous at* $p \in X$ *iff* $f^*(\mu(p)) \subset \mu_\alpha(f(p))$.

PROOF. (i) Recall that a map $f: X \to Y$ is θ-continuous at $p \in X$ iff for $f(p) \in G \in T$ there exists some $H \in \tau$ such that $f(\mathrm{cl}_X H) \subset \mathrm{cl}_Y G$. Thus $<\{\mathrm{cl}_Y G \mid f(p) \in G \in T\}> \subset <f[\overline{G(p)}]>$. Consequently, Nuc $f[\overline{G(p)}] \subset \mu_\theta(f(p))$. Now $f^*(\mu_\theta(p)) \subset$ Nuc Fil $(f^*(\mu_\theta(p)) =$ Nuc $f[\overline{G(p)}]$ implies that $f^*(\mu_\theta(p)) \subset \mu_\theta(f(p))$.

Conversely, if $f^*(\mu_\theta(p)) \subset \mu_\theta(f(p))$, then Nuc Fil $(f^*(\mu_\theta(p))) \subset \mu_\theta(f(p))$ implies that $<\{\mathrm{cl}_Y G [f(p) \in G \in T\}> \subset <f[G(p)]>$, then this implies that for each cl_YG such that $f(p) \in G \in T$ there exists some $\mathrm{H} \in G(p)$ such that $f(H) \subset \mathrm{cl}_Y$G and the result follows.

(ii), (iii), (iv). Left to the reader.

The concepts of the almost-open map can also be easily characterized.

THEOREM 5.2. *Let* $f:(X,\tau)\to(Y,T)$ *and* $p\in X$.

(i) *If* $\mu(f(p))\subset f^*(\mu_\alpha(p))$, *then* f *is almost-open at* $p\in X$.

(ii) *If* $^*M$ *is highly saturated and* f *is almost-open at* $p\in X$, *then* $\mu(f(p))\subset f^*(\mu_\alpha(p))$.

PROOF. Simply consider the semiregularization and use the classical results found in [9], [12].

REMARK. The reader should be able to minic the classical nonstandard results about open maps to obtain interesting characterizations for other generalizations of the open map.

6. Applications

We easily establish by nonstandard means many standard results. Of course, we shall only exhibit a few such conclusions leaving many others to the reader. It is now obvious that a nearly-compact semiregular space is compact. Recall that a map $f:X\to Y$ is weakly-open at $p\in X$ if for each open neighborhood G of p there exists an open neighborhood H of $f(p)$ such that $H\subset f(\mathrm{cl}_X G)$. Clearly, open at $p\in X$ implies almost-open at $p\in X$ implies weakly-open at $p\in X$, since $\mu(p)\subset\mu_\alpha(p)\subset\mu_\theta(p)$. The following result is an immediate consequence of our prior theorems.

THEOREM 6.1. *Let* X *be regular [resp. semiregular, almost-regular] at* $p\in X$. *Then* $f:X\to Y$ *is weakly-open [resp. almost-open, weakly-open] at* $p\in X$ *iff* f *is open [resp. open, almost-open] at* $p\in X$.

THEOREM 6.2. *An almost-regular* T_2 *space is Urysohn.*

PROOF. Theorems 4.1, 4.2, 4.3.

THEOREM 6.3. *A space is nearly-compact* T_2 *iff* $\{\mu_\alpha(p)\mid p\in X\}$ *is a partition for* X^*.

PROOF. Theorems 3.1, 4.3.

THEOREM 6.4. *A nearly-compact* T_2 *space is almost-regular* T_2.

PROOF. Let $p\in X, q\in X^*$ such that $q\notin\mu_\alpha(p)$. Then there exists some p_1 such that $q\in\mu_\alpha(p_1)$, $\mu_\alpha(p)\cap\mu_\alpha(p_1)=\phi$. Now apply 4.2 and 4.3. Of course, this also follows from the obvious fact that the semiregularization is compact T_2. Thus T_3. Interpreting this in the original topology also gives the result.

The previous conclusions imply that a nearly-compact T_2 space is Urysohn and an almost-regular quasi-H-closed space is nearly-compact. Clearly, a space X is H-closed Urysohn iff $\{\mu_\theta(p)\mid p\in X\}$ partitions X^*. Indeed, an H-closed Urysohn space is almost-regular. Consequently, a space is nearly-compact T_2 iff it is H-closed, Urysohn. In previous papers [3], [4] we have shown that Baire, quasi-H-closed, H(i) semiregular, U(i) and R(i) extensions for an arbitrary space exist as subsets of X^* as well as normal compactifications for a completely regular (not necessarily T_1) space and the Wallman type compactifications for a Tychonoff space. Among these extensions of a space X, Stroyan [17] has shown that the Wallman type compactifications may be viewed as quotients of X^*. It has become increasingly important to determine those objects which can be considered as quotients, subquotients, or subsets of the nonstandard extension of the base space. Most recently it has been shown that the projective cover of a compact T_2 space X may be considered a subquotient of X^* [2]. More specifically, it is implicit within Robinson's results [12] that all compact T_2 spaces X can be considered as quotients of their enlargements X^*. We now easily generalize this fundamental result.

THEOREM 6.5. *Every H-closed Urysohn space X is homeomorphic to a quotient of X^*.*

PROOF. For each $p\in X$, we define the map $f:X^*\to X$ by letting $f(\mu_\theta(p))=p$. This is well defined and onto since $\{\mu_\theta(p)\mid p\in X\}$ partitions X^* and $\mu_\theta(p)\cap X=\{p\}$, by 4.3(i), for each $p\in X$. The result follows in the usual manner by letting X^* carry the weak topology induced by f.

Since we know that a non-compact Tychonoff space can be densely embedded into a non-compact H-closed Urysohn space, then we have that each non-compact Tychonoff space X can be densely embedded into a non-compact quotient of X^*, where this quotient has all the properties stated by Porter and Thomas in [11].

We shall now use the previous conclusions to obtain an important new result concerning the *-proper maps defined by Rudolf in [13].

THEOREM 6.6. *Let D be a dense and open subspace in (X,τ). Assume that $X-D$*

*is a discrete space and the space Y is H-closed. Then every θ-continuous *-proper map $f:D\to Y$ has a unique weakly-θ-continuous extension to X.*

PROOF. Assume that $^*M$ is highly saturated and that $f:D\to Y$ is *-proper. Then theorem 2.4 in [13] implies that the open filter $\mathscr{F}=<\{G|G$ is open in Y and $G\supset f(U_r\cap D)$, $r\in U_r\in\tau\}>$ where $r\in X-D$ has a unique cluster point. Let $F=\{f(G_r\cap D)|G_r\in G(r)\}$, where $r\in X-D$. Clearly, for each $r\in X-D$ we have that Nuc $F\subset$Nuc $\mathscr{F}$. However, since Y is H-closed, then there exists some $q\in Y$ such that $\mu_\theta(q)\cap$Nuc $F\neq\phi$. Since f is *-proper, this implies that q is unique. Consequently, St_θ(Nuc F)$=\{p|p\in X$ and $\mu_\theta(p)\cap$Nuc $F\neq\phi\}=\mathrm{St}_\theta(f^*(\mu(r)\cap D^*))=$a singleton set.

We now construct the extension $F:X\to Y$ of f. Let $r\in X-D$. Since $\mathrm{St}_\theta(f^*(\mu(r)\cap D^*))=$a singleton set and Y is H-closed then there exists a unique $p_r\in Y$ such that $f^*(\mu(r)\cap D^*)\subset\mu_\theta(p_r)$. Observe that $\mu(r)\cap(X-D)^*=\{r\}$ since $X-D$ is discrete and that $\mu_\theta(p_r)\cap Y=\{p_r\}$. Consequently, we define for each $r\in X-D$, $F(r)=p_r$. Then for each $p\in D$, let $F(p)=f(p)$.

In order to show that F is weakly-θ-continuous, we first let $r\in X-D$. Then $F^*(\mu(r))=F^*((\mu(r)\cap(X\text{-}D)^*)\ \cup(\mu(r)\cap D^*))=F^*(r)\cup f^*(\mu(r)\cap D^*)\subset\mu_\theta(F(r))$. Thus F is weakly-θ-continuous on $X-D$. We now let $p\in D$. Then $F^*(\mu(p))=F^*(\mu(p)\cap D^*)=f^*(\mu(p)\cap D^*)\subset\mu_\theta(f(p))$ since D is open in X and f is weakly-θ-cont inuous. Evidently, F is unique and weakly-θ-continuous on X.

COROLLARY 6.1. *Let D be a dense and open subspace in (X,τ). Let Y be H-closed Urysohn. Then any *-proper almost-continuous $f:D\to Y$ has a unique almost-continuous extension $F:X\to Y$.*

Many extensions (X,τ) of D satisfy the hypothesis of theorem 6.6 or corollary 6.1. For example, the quasi-H-closure kD of D. In the case of a Hausdorff kD, any almost-continuous $f:D\to Y$ is *-proper.

For certain specialized spaces, the θ and α-monads are capable of completly characterizing uniqueness of cluster points as the next two results indicate.

THEOREM 6.7. *If X is quasi-H-closed and $\mathscr{F}$ is an open filter with a unique cluster point $p\in X$, then* Nuc$\mathscr{F}\subset\mu_\theta(p)$.

PROOF. Left of the reader.

THEOREM 6.8. *If X is Urysohn, $\mathscr{F}$ is an open filter on X and* Nuc$\mathscr{F}\subset\mu_\theta(p)$

for some $p\in X$, then p is a unique cluster point for $\mathscr{F}$.

PROOF. Left to reader.

As a final result, we shall give a criterion which will determine whether a non-empty set $S\subset X$ contains the closure of a non-empty open set or a non-empty regular-open set

THEOREM 6.9. *Let $\phi\neq S\subset X$. Then*

(i) *S contains the closure of a non-empty open set iff $\mu_\theta(p)\subset S^*$ for some $p\in S$;*

(ii) *S contains a non-empty regular-open set iff $\mu_\alpha(p)\subset S^*$ for some $p\in S$.*

PROOF. We shall only prove (i) since (ii) is similar. Assume that S contains a non-empty cl G, where G is open. Thus $\mu_\theta(p)\subset(\text{cl } G)^*\subset S^*$.

Conversely, assume that $\mu_\theta(p)\subset S^*$ for some $p\in S$. Then there exists an infinitesimal *element E $^*\!\in\overline{G(p)}$. The sentence in L

$$\exists x\,[x\in\overline{G(p)}]\wedge[x\subset S]$$

holds in $^*M$; hence in M and the result follows from its interpretation in M.

We shall let the reader obtain other interesting results; especially those involving nearly-compact, quasi-H-closed spaces and various types of maps. Clearly, the above results amply indicate the economy of effort inherent in proving standard results by nonstandard methods and certainly give further insight into the concepts concerned.

Finally, it is sincerely hoped that the reader will find as much application for the θ and α-monads as has been found for the monad of Robinson.

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REFERENCES

[1] S.V. Fomin, *Extensions of topological spaces*, Dokl. Akad. Nauk SSSR 32 (1941), 114—117.

[2] H.Gonshor, *Projective covers as subquotients of enlargements*, Israel J. Math. 14 (1973), 257—261.

[3] R.A. Herrmann, *Nonstandard topological extensions*, Bull. Austral. Math. Soc. 13 (1975), 269—290.

[4] ________, *H*(i) *semiregular, U*(i) *and R*(i) *extensions* (to appear).

[5] N. Levine, *Strong continuity*, Amer. Math. Monthly 67 (1960), 269.

[6] ________, *A decomposition of continuily in topological spaces*, Amer. Math. Monthly 68 (1961), 44—46.

[7] C.T. Liu, *Absolutely closed spaces*, Trans. Amer. Math. Soc. 130 (1968), 86—104.

[8] N.A.J. Luxemburg, *A general theory of monads*, in Applications of Model Theory to Algebra, Analysis and Probability, Holt, Rinehart and Winston, 1969.

[9] M. Machover and J. Hirschffld, *Lectures on Non-Standard Analysis*, Lecture Notes in Math. No.94, Springer-Verlag 1969.

[10] A. Mathur, *Some weaker forms of compactness in General Topology and its Relations to Modern Analysis and Algebra III*, Academic Press, 1971.

[11] J.R. Porter and J.D. Thomas, *On H-closed and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. 138 (1969), 159—170.

[12] A. Robinson, *Non-Standard Analysis*, North-Holland, 1966.

[13] L. Rudolf, *Extending maps from dense subspaces*, Fund. Math. 74 (1972), 171—190.

[14] M.K. Singal and S.P. Arya, *On almost-regular spaces*, Glasnik Math. 4 (24) (1969), 89—99

[15] M.K. Singal and A. Mathur, *On nearly-compact spaces*, Boll. U.M.I. 4 (1969), 702—710.

[16] M.K. Singal and A. Rani, *On almost-continuous mappings*, The Yokohama Math. J. 16 (1968), 63—73.

[17] K.D. Stroyan, *Additional remarks on the theory of monads*, in Contributions to Non-Standard Analysys, ed Luxemburg and Robinson, North-Holland, 1972.