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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF WEAKLY COUPLED PARABOLIC SYSTEMS WITH UNBOUNDED COEFFICIENTS

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1. Introduction

Let E^n be the n-dimensional Euclidean space whose generic point x is denoted by its coordinates $(x_1,\ldots,x_n)$ and let t be the time variable on the real line. The distance of a point $x\in E^n$ to the origin is defined by $|x|=\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Consider the system of parabolic differential equations

$$(1) \qquad L^{\alpha}[u^{\alpha}]+\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta}=f^{\alpha}(x,t), \qquad \alpha=1,2,\ldots,N,$$

where each L^{α} stand for the parabolic operator

$$L^{\alpha}=\sum_{i,j=1}^{n} a_{ij}^{\alpha}(x,t)\frac{\partial^2}{\partial x_i\,\partial x_j}+\sum_{i=1}^{n} b_i^{\alpha}(x,t)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t}.$$

Each equation of (1) contains derivatives of just one component of the unknown functions $u^1(x,t)$, $u^2(x,t),\ldots,$ $u^N(x,t)$, and the system (1) is coupled only in the terms which are not differentiated; so that a system of this form is said to be weakly coupled [1].

Recently, Kusano-Kuroda-Chen [2] investigated the asymptotic behavior for $t\to\infty$ of solutions of the Cauchy problem for the weakly coupled parabolic systems $L^{\alpha}[u^{\alpha}]+\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta}=0$, $\alpha=1,\ldots,N$, with unbounded coefficients.

In [4], Chabrowski discussed the behavior of decay for $t\to\infty$ of solution of a single parabolic equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=f(x,t)$$

with bounded coefficients in E^{n+1}.

In this note, we extend the Chabrowski's result to the system (1) with unbounded coefficients.

2. Maximum principles

In this section we are concerned with the weakly coupled system of parabolic inequalities

$$L^\alpha[u^\alpha]+\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)\,u^\beta\geq 0,\qquad \alpha=1,\dots,N, \tag{2}$$

where L^α as described in section 1.

The following two maximum principles due to Kusano-Kuroda-Chen [2] will be important in the later treatment.

LEMMA 1. [2]. *Suppose the coefficients of* (2) *in* $E^n\times[0,\infty)$ *satisfy the following inequalities*

$$(3)\quad\begin{cases} 0\leq \sum_{i,j=1}^{n} a_{ij}^\alpha(x,t)\xi_i\xi_j\leq K_1\,[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2\\ |b_i^\alpha(x,t)|\leq K_2(|x|^2+1)^{\frac{1}{2}},\quad i=1,\dots,n,\\ c^{\alpha\beta}(x,t)\geq 0,\quad \alpha\neq\beta,\end{cases}$$

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)\leq K_3\,[\log([x|^2+1)+1]^\lambda\;(|x|^2+1)^\mu$$

for all real n-vectors $\xi=(\xi_1,\dots,\xi_n)$ *and* $\alpha,\beta=1,\dots,N$, *where* $K_1>0$, $K_2\geq 0$, $K_3>0$, $\mu>0$ *and* λ *are constants. Let* $u^\alpha(x,t)$, $\alpha=1,\dots,N$, *satisfy* (2) *in* $E^n\times(0,\infty)$ *with the properties*:

$$u^\alpha(x,0)\leq 0\ \textit{for}\ x\in E^n,\quad \alpha=1,\dots,N,$$

and

$$u^\alpha(x,t)\leq M\,\exp\{k[\log(|x]^2+1)+1]^\lambda(|x|^2+1)^\mu\}\ \textit{for}\ (x,t)\ E^n\times(0,\infty)$$

where M and k are some positive constants. Then $u^\alpha(x,t)\leq 0$ *in* $E^n\times(0,\infty)$, $\alpha=1,2,\dots,N$.

LEMMA 2. [3]. *Assume that the coefficients of* (2) *in* $E^n\times(0,\infty)$ *satisfy the inequalities*

$$0\leq\sum_{i,j=1}^{n} a_{ij}^\alpha(x,t)\xi_i\xi_j\leq K_1[\log(|x|^2+1)+1]^{2-\lambda}(|x|^2+1)|\xi|^2$$

$$(4)\quad |b_i^\alpha(x,t)|\leq K_2[\log(|x|^2+1)+1](|x|^2+1)^{\frac{1}{2}},\quad i=1,2,\dots,n,$$

$$c^{\alpha\beta}(x,t)\geq 0\ \textit{for}\ \alpha\neq\beta$$

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)\leq K_3[\log(|x|^2+1)+1]^\lambda,$$

for all real n-vectors $\xi=(\xi_1,\dots,\xi_n)$ *and* $\alpha,\beta=1,2,\dots,N$, *where* $K_1>0$, $K_2>0$, $K_3>0$, *and* $\lambda>1$ *are constants. Let* $u^\alpha(x,t)$, $\alpha=1,\dots,N$, *satisfy* (2) *in* $E^n\times(0,\infty)$ *with the properties*:

$$u^\alpha(x,0)\leq 0 \text{ for } x\in E^n,\ \alpha=1,\dots,N,$$

and

$$u^\alpha(x,t)\leq M\exp\{k[\log(|x|^2+1)+1]^\lambda\} \text{ in } E^n\times(0,\infty)$$

for some positive constants M and k, $\alpha=1,\dots,N$. *Then* $u^\alpha(x,t)\leq 0$ *in* $E^n\times(0,\infty)$, $\alpha=1,\dots,N$.

3. Exponential decay of solutions as $t\to\infty$

By a solution of (1) we mean a system of N real valued functions $u^\alpha(x,t)$, $\alpha=1,\dots,N$, which are continuous in $E^n\times[0,\infty)$, continuously differentiable once with respect to t and twice with respect to x in $E^n\times(0,\infty)$ and satisfy the system (1) in $E^n\times(0,\infty)$.

THEOREM 1. *Suppose the coefficients of* (1) *satisfy the condition* (3) *and* $\sum_{\beta=1}^N c^{\alpha\beta}(x,t)\leq -K_3$, *where* $\alpha=1,\dots,N$, $K_3>0$ *is a constant. Let* $u^\alpha(x,t)$, $\alpha=1,\dots,N$, *be a bounded solution of* (1). *If* $\lim_{t\to\infty} f^\alpha(x,t)=0$, $\alpha=1,\dots,N$, *uniformly with respect to* $x\in E^n$, *then* $\lim_{t\to\infty} u^\alpha(x,t)=0$, $\alpha=1,\dots,N$, *uniformly with respect to* $x\in E^n$.

PROOF. Let ε be an arbitrary positive number. We see easily that there exists a positive constant δ such that

$$|f^\alpha(x,t)|\leq\varepsilon,\ \alpha=1,\dots,N,$$

for $x\in E^n$ and $t\geq\delta$. Put

$$M^\alpha=\sup_{(t,t)\in E^{n+1}}|u^\alpha(x,t)|,\ \alpha=1,\ \dots,\ N.$$

We introduce the auxiliary functions

$$w_\pm^\alpha(x,t)=-2\frac{\varepsilon}{K_3}-M^\alpha e^{-r(t-s)}\pm u^\alpha(x,t),\ \alpha=1,\dots,N,$$

where r is a positive constant such that $0<r<K_3$. Hence

$$L^\alpha[w_\pm^\alpha]+\sum_{\beta=1}^N c^{\alpha\beta}(x,t)u^\beta=-\frac{2\varepsilon}{K_3}\sum_{\beta=1}^N c^{\alpha\beta}(x,t)-M^\alpha e^{-r(t-\delta)}\sum_{\beta=1}^N c^{\alpha\beta}(x,t)$$

$$-rM^{\alpha}e^{-r(t-\delta)}\pm f^{\alpha}(x,t)$$

$$\geq \varepsilon + M^{\alpha}e^{-r(t-\delta)}(K_3-r)>0,\ \alpha=1,\dots,N.$$

for $x\in E^n$ and $t>\delta$. Moreover

$$w_{\pm}^{\alpha}(x,\delta)=-2\frac{\varepsilon}{K_3}-M^{\alpha}+u^{\alpha}(x,\delta)<0,\ \alpha=1,\dots,N$$

for $x\in E^n$. From Lemma 1, we see

$$w_{\pm}^{\alpha}(x,t)\leq 0,\qquad \alpha=1,\dots,N$$

for $x\in E^n$ and $t\geq\delta$. Hence

$$-2\frac{\varepsilon}{K_3}-M^{\alpha}e^{-r(t-\delta)}\leq u^{\alpha}(x,t)\leq 2\frac{\varepsilon}{K_3}+M^{\alpha}e^{-r(t-\delta)},$$

for $x\in E^n$, $t\geq\delta$ and $\alpha=1,\dots,N$. Therefore

$$-\frac{2\varepsilon}{K_3}\leq \liminf_{t\to\infty} u^{\alpha}(x,t)\leq \limsup_{t\to\infty} u^{\alpha}(x,t)\leq\frac{2\varepsilon}{K_3}$$

which proves our theorem.

Similarly, using Lemma 2, we can prove the following.

THEOREM 2. *Suppose the coefficients of* (1) *satisfy the condition* (4) *and* $\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)\leq -K_3$, *where* $\alpha=1,\dots,N$, $K_3>0$ *is a constant. Let* $u^{\alpha}(x,t)$, $\alpha=1,\cdots,N$, *be a bounded solution of* (1). *If* $\lim_{t\to\infty} f^{\alpha}(x,t)=0$, $\alpha=1,\dots,N$, *uniformly with respect to* $x\in E^n$, *then* $\lim_{t\to\infty} u^{\alpha}(x,t)=0$ *uniformly with respect to* $x\in E^n$.

REMARK. In the case $\lambda=0$, $\mu=1$, $N=1$, Theorem 1 of [4] is a special case of our Theorem 1.

EXAMPLE. The system

$$\Delta u^1-(|x|^2+2)u^1+u^2-\frac{\partial u^1}{\partial t}=-ne^{-\frac{|x|^2}{2}-t}$$

$$\Delta u^2+u^1-(|x|^2+2)u^2-\frac{\partial u^2}{\partial t}=-ne^{-\frac{|x|^2}{2}-t}$$

which has a solution $u^1(x,t)=u^2(x,t)=e^{-\frac{|x|^2}{2}-t}$, where Δ is the n-dimensional Laplace operator. Obviously $\lim_{t\to\infty} u^1(x,t)=\lim_{t\to\infty} u^2(x,t)=0$. For this system, $\lambda=0$,

$$\mu=1,\ \ K_1=1,\ \ K_2=0,\ \ K_3=1,\ \lim_{t\to\infty} f^i(x,t)=-n\lim_{t\to\infty} e^{-\frac{|x|^2}{2}-t}=0 \text{ for } i=1,2.$$

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