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INFINITE SERIES OF KAMPE DE FERIET'S DOUBLE HYPERGEOMETRIC FUNCTIONS OF HIGHER ORDER

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1. Introductory

Kampe de Feriet's [1] introduced the double hypergeometric function of higher order (i.e. with more parameters) in two variables, namely

$$F\left(\begin{array}{c|c|c} \mu & \alpha_1,\dots,\alpha_\mu & \\ \nu & \beta_1, \beta_1';\dots;\beta_\nu, \beta_\nu' & \\ p & \gamma_1,\dots,\gamma_p & x,\ y \\ \sigma & \delta_1, \delta_1';\dots;\delta_\sigma, \delta_\sigma' & \end{array}\right)$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\alpha_j;m+n)\prod_{j=1}^{\nu}\{(\beta_j;m)(\beta_j';n)\}}{\prod_{j=1}^{p}(\gamma_j;m+n)\prod_{j=1}^{\sigma}\{(\delta_j;m)(\delta_j';n)\}}\ \frac{x^m\ \ y^n}{(1;m)\ (1;n)}; \tag{1}$$

where $\mu+\nu\leq p+\sigma+1$.

For the definition and properties of this function the reader is referred to [1], pp. 147—176. For special values of the parameters μ,ν,p,σ, the function (1) reduces to the four double hyhergeometric functions of Appel.

Thus we have ([1], p. 14):

$$F\left(\begin{array}{c|cc|c} 1 & \alpha & & \\ 1 & \beta_1, & \beta_1' & \\ 0 & \cdot\cdot & \cdot\cdot & x,\ y \\ 1 & \delta_1, & \delta_1' & \end{array}\right)=F^{[2]}[\alpha;\beta_1, \beta_1';\delta_1, \delta_1';x, y]; \tag{2}$$

$$F\left(\begin{array}{c|c|c} 2 & \alpha_1, \alpha_2 & \\ 0 & \cdots\cdots & \\ 0 & \cdots\cdots & x,\ y \\ 1 & \delta_1, \delta_1' & \end{array}\right)=F^{[4]}[\alpha_1, \alpha_2;\delta_1, \delta_1';x, y]. \tag{3}$$

Also it is easily seen that

$$F\begin{pmatrix} 0 & \cdots\cdots & \\ \nu & \beta_1,\ \beta_1';\dots;\beta_\nu,\ \beta_\nu' & \\ 0 & \cdots\cdots & x,\ y \\ \sigma & \delta_1,\ \delta_1';\dots;\delta_\sigma,\ \delta_\sigma' & \end{pmatrix} = {}_\nu F_\sigma\begin{pmatrix} \beta_1,\dots,\beta_\nu;x \\ \delta_1,\dots,\delta_\sigma \end{pmatrix} {}_\nu F_\sigma\begin{pmatrix} \beta_1',\dots,\beta_\nu';y \\ \delta_1',\dots,\delta_\sigma' \end{pmatrix} \quad (4)$$

and

$$F\begin{pmatrix} \mu & \alpha_1,\dots,\alpha_\mu & \\ 0 & \cdots\cdots & x,\ x \\ p & \gamma_1,\dots,\gamma_p & \\ 1 & \delta_1,\delta'_1 & \end{pmatrix}$$

$$= {}_{\mu+2}F_{p+3}\begin{pmatrix} \alpha_1,\dots,\alpha_\mu,\ \frac{1}{2},\ \ \frac{1}{2}\delta_1+\frac{1}{2}\delta_1'-\frac{1}{2},\ \ \frac{1}{2}\delta_1+\frac{1}{2}\delta_1';4x \\ \gamma_1,\dots,\gamma_p,\delta_1,\ \delta'_1,\ \delta_1+\delta_1'-1; \end{pmatrix}; \quad (5)$$

where $\mu \leq p+2$ and $|x| < \frac{1}{4}$ when $\mu = p+2$.

J. Burchnall and T. W. Chaundy [2] gave an extensive list of expansions of Appell's double hypergeometric functions, they introduced a certain type of differential operator and deduced their results by an application of these operators. My method in deriving the main theorem is striaghtforward and is based on series derangement. The main theorem will be stated and proved in § 2; while particular cases will be deduced in § 3. It may be noted that the constants and the parameters are such that the functions involved exist. Dougall's theorem [Proc. Edin. Math. Soc. XXV, 1906, 10] is required in the proof, namely;

$${}_5F_4\begin{bmatrix} a,1+\frac{1}{2}a,\ c,\ d,\ e;\ 1 \\ \frac{1}{2}a,\ 1+a-c,\ 1+a-d,\ 1+a-e \end{bmatrix}$$

$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\ \Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-d)\Gamma(1+a-c-e)}, \quad (6)$$

where $\mathrm{R}(a-c-d-e) > -1$

2. The maim theorem

The expansion to be established is

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j;\ 2r)\ \prod_{j=1}^{\nu}\{(\beta_j;r)(\beta_j';r)\}\ (a;r)^2(a-b;r)}{r!\ \prod_{j=1}^{p}(\gamma_j;2r)\ \prod_{j=1}^{\sigma}\{(\delta_j;r)(\delta_j';r)\}\ (a;2r)(b;\ r)}$$

$$\times\frac{(xy)^r}{(a+r-1;r)}\;F\left(\begin{array}{c|l|c}\mu & \alpha_1+2r,\ldots,\alpha_\mu+2r & \\ \nu+1 & \beta_1+r,\ \beta'_1+r;\ldots;\beta_\nu+r,\ \ \beta'_\nu+r;a+r,a+r & xy\\ p & \gamma_1+2r,\ \ldots,\ \gamma_p+2r & \\ \delta+1 & \delta_1+r,\ \delta'_1+r;\ldots;\delta_\sigma+r,\ \delta'_\sigma+r;a+2r,\ a+2r & \end{array}\right)$$

$$=F\left(\begin{array}{c|l|c}\mu+1 & \alpha_1,\ldots,\alpha_\mu,b & \\ \nu+1 & \beta_1,\beta'_1;\ldots;\beta_\nu,\beta'_\nu;a;a & x,\ y\\ p+1 & \gamma_1,\ldots,\gamma_p,a & \\ \sigma+1 & \delta_1,\delta'_1;\ldots;\delta_\sigma,\delta'_\sigma;b,b & \end{array}\right). \tag{7}$$

where $\mu+\nu\leq p+\sigma+1$ and $(\alpha;r)=\Gamma(\alpha+r)/\Gamma(\alpha)$, $(\alpha,0)=1$.

PROOF. When $\mu+\nu\leq p+\sigma+1$, then from (1), the left side of (7) is equal to

$$\sum_{r=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\alpha_j;\,2r)\prod_{j=1}^{\nu}\{(\beta_j;r)\,(\beta'_j;r)\}\,(a;r)^2(a-b;\,r)}{r!\,\prod_{j=1}^{p}(\gamma_j;2r)\prod_{j=1}^{\sigma}\{(\delta_j;\,r)\,(\delta'_j,\,r)\}\,(a;2r)(b;r)}\,\frac{(xy)^r}{(a+r-1;r)}$$

$$\times\frac{\prod_{j=1}^{\mu}(\alpha_j+2r;\,m+n)\prod_{j=1}^{\nu}\{(\beta_j+r;\,m)\,(\beta'_j+r;n)\}\{(a+r;m)(a+r;n)}{\prod_{j=1}^{p}(\gamma_j+2r;m+n)\prod_{j=1}^{\sigma}\{(\delta_j+r;m)\,(\delta'_j+r;n)\}\,(a+2r;m)(a+2r;n)}\,\frac{x^m\ y^n}{m!\ n!}$$

Here write $m=p-r$, $n=q-r$. Change the order of summation putting the first summation last and the last expression becomes

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\prod_{j=1}^{\mu}(\alpha_j;p+q)\prod_{j=1}^{\nu}\{(\beta_j,p)(\beta'_j;q)\}}{\prod_{j=1}^{p}(\gamma_j;\,p+q)\prod_{j=1}^{\sigma}\{(\delta_j;p)(\delta'_j;q)\}}\,\frac{x^p\,y^q}{p!\ q!}\,{}_5F_4\left[\begin{matrix}a-1,\ \frac{a}{2}+\frac{1}{2},a-b,\,-p,\,-q;1\\ \frac{a}{2}-\frac{1}{2},\ b,\ a+p,\ a+q\end{matrix}\right]$$

Now sum the ${}_5F_4$ by Dougall's theorem (6), apply (1) again and so obtain the right hand side of (7). Thus (7) is proved.

3. Particular cases

In (7) taking $\mu=1,\nu=p=\sigma=0$, applying (2)

$$\sum_{r=0}^{\infty}\frac{(a;r)^2(a-b;r)(\alpha;2r)}{r!(a;2r)(a+r-1;r)(b;r)}\ (xy)^r\ F^{[2]}\ [\alpha+2r;a+r,a+r;a+2r,a+2r;x,y]$$

$$=F\left(\begin{array}{c|c|c}2 & \alpha,\ b & \\ 1 & a,\ a & x,\ y\\ 1 & a & \\ 1 & b,\ b & \end{array}\right). \tag{8}$$

In (7) taking $\mu=2, \nu=1, p=0, \sigma=1,$ with $\delta_1=\delta'_1=a$, applying (3).

$$\sum_{r=0}^{\infty} \frac{(\alpha_1;2r)(\alpha_2;2r)(a;r)^2(a-b;r)}{r!(a;r)\ (a;r)\ (a;2r)\ (a+r-1;r)\ (b;r)} (xy)^r$$
$$\times F^{[4]}\ [\alpha_1+2r,\ \alpha_2+2r;\ a+2r, a+2r;\ x, y]$$
$$=F\left(\begin{array}{c|c|c} 3 & \alpha_1, \alpha_2, b & \\ 0 & \cdots\cdots\cdots & x,\ y \\ 1 & a & \\ 1 & b,\ \ b & \end{array}\right). \tag{9}$$

In (7) taking $\mu=p=0$, applying (4)

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\nu}\{(\beta_j;r)(\beta'_j;r)\}\ (a-b;r)(a;r)^2}{r!\ \prod_{j=1}^{\sigma}\{(\delta_j;r)(\delta'_j;r)!\ (a;2r)\}\ (a+r-1;r)(b;r)}$$
$$\times (xy)^r{}_{\nu+1}F_{\sigma+1}\left(\begin{matrix}\beta_1+r, \ldots, \beta_\nu+r,\ a+r:\ x\\ \delta_1+r, \ldots, \delta_\sigma+r,\ a+2r\end{matrix}\right){}_{\nu+1}F_{\sigma+1}\left(\begin{matrix}\beta'_1+r,\ \ldots,\ \beta'_\nu+r,\ a+r:\ y\\ \delta'_1+r,\ \ldots,\ \delta'_\sigma+r,\ a+2r\end{matrix}\right)$$
$$=F\left(\begin{array}{c|c|c} 1 & b & \\ \nu+1 & \beta_1, \beta'_1, \ldots, \beta_\nu, \beta'_\nu,\ a,\ a & x,\ y \\ 1 & a & \\ \sigma+1 & \delta_1, \delta'_1, \ldots, \delta_\sigma, \delta'_\sigma, b, b & \end{array}\right) \tag{10}$$

where $|x|<1, |y|<1$.

In (10) taking $\nu=\sigma=n$, I get the summation of generalized Whittaker's function

$$\sum_{i=0}^{\infty} \frac{\prod_{j=1}^{n}\{(\beta_j;r)(\beta'_j;r)\}(a;r)^2(a-b;r)}{r!\ \prod_{j=1}^{n}\{(\delta_j;r)(\delta'_j;r)\}\ (a;2r)(a+r-1;r)(b;r)} (xy)^r$$
$$\times {}_{n+1}F_{n+1}\left(\begin{matrix}\beta_1+r, \ldots, \beta_n+r,\ a+r:\ x\\ \delta_1+r,\ \ldots,\ \delta_n+r,\ a+2r\end{matrix}\right){}_{n+1}F_{n+1}\left(\begin{matrix}\beta'_1+r,\ \ldots,\ \beta'_n+r,\ a+r:y\\ \delta'_1+r,\ \ldots,\ \delta'_n+r,\ a+2r\end{matrix}\right)$$
$$=F\left(\begin{array}{c|c|c} 1 & b & \\ n+1 & \beta_1, \beta'_1,\ \ldots,\ \beta_n,\ \beta'_n,\ a,\ a & x,\ y \\ 1 & a & \\ n+1 & \delta_1,\ \delta'_1,\ \ldots,\ \delta_\sigma,\ \delta'_\sigma,\ b,\ b & \end{array}\right). \tag{11}$$

In (7) taking $\nu=0,\ \sigma=1,$ with $\delta_1=\delta'_1=a$, applying (5).

$$\sum_{r=0}^{\infty} \frac{\prod_{j=1}^{\mu}(\alpha_j;2r)(a-b;r)}{r!\prod_{j=1}^{p}(\gamma_j;2r)(a;2r)(a+r-1;r)(b;r)} x^{2r}$$

$$\times {}_{\mu+1}F_{p+2}\left(\begin{matrix}\alpha_1+2r,\ldots,\alpha_\mu+2r, a+2r-\frac{1}{2}: 4x\\ \gamma_1+2r,\ldots,\gamma_p+2r, a+2r, 2a+4r-1\end{matrix}\right)$$

$$=F\left(\begin{matrix}\mu+1\\0\\p+1\\1\end{matrix}\left|\begin{matrix}\alpha_1,\ldots,\alpha_\mu, b\\ \cdots\cdots\cdots\cdots\\ \gamma_1,\ldots,\gamma_p, a\\ b, b\end{matrix}\right| x,\ x\right), \tag{12}$$

where $\mu \leq p+2$ and $|x| < \frac{1}{4}$ when $\mu = p+2$.

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