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ON THEOREMS FOR THREE VARIABLES ANALOGOUS TO WATSON'S AND WHIPPLE'S THEOREM

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1. Introduction

The triple hypergeometric function of superior order is defined as [2, p. 14]

$$(1.1)\quad F\left[\begin{matrix}(a_A);(b_{B_1}),(b'_{B_2}),(b''_{B_3}),(c_{C_1}),(c'_{C_2}),(c''_{C_3});\\ (d_D);(e_{E_1}),(e'_{E_2}),(e''_{E_3}),(f_{F_1}),(f'_{F_2}),(f''_{F_3});\end{matrix}\ x,y,z\right]$$

$$=\sum_{m,n,p=0}^{\infty}\frac{\prod_{j=1}^{A}(a_j,m+n+p)\prod_{j=1}^{B_1}(b_j,m+n)\prod_{j=1}^{B_2}(b_j',m+p)}{\prod_{j=1}^{D}(d_j,m+n+p)\prod_{j=1}^{E_1}(e_j,m+n)\prod_{j=1}^{E_2}(e_j',m+p)}$$

$$\times\frac{\prod_{j=1}^{B_3}(b_j'',n+p)\prod_{j=1}^{C_1}(c_j,m)\prod_{j=1}^{C_2}(c_j',n)\prod_{j=1}^{C_3}(c_j'',p)\,x^m y^n z^p}{\prod_{j=1}^{E_3}(e_j'',n+p)\prod_{j=1}^{F_1}(f_j,m)\prod_{j=1}^{F_2}(f_j',n)\prod_{j=1}^{F_1}(f_j'',p)m!\,n!\,p!}$$

where $A, B_1, B_2, B_3, C_1, C_2, C_3, D, E_1, E_2, E_3, F_1, F_2$ and F_3 are positive integers m, n, p takes all positive integral values from 0 to ∞ and the parameters satisfy the following conditions:

$$A+B_1+B_2+C_1\leq D+E_1+E_2+F_1+1$$
$$A+B_1+B_3+C_2\leq D+E_1+E_3+F_2+1$$
$$A+B_2+B_3+C_3\leq D+E_2+E_3+F_3+1$$

and

$$(a_A)=(a_1,a_2,\dots,a_A)$$
$$(a,m)=(a)_m=a(a+1),\dots,(a+m-1)$$
$$(a,0)=1$$

In this paper we give interesting special cases of this function, which is the triple sums analogous to Watson's and Whipple's theorem.

2. We prove the following theorems

(a) Watson's theorem for three variables

$$(2.1)\quad F\left[\begin{matrix} a_1, a_2;\ -,-,-,\ b, b', b''; \\ \frac{1}{2}+\frac{1}{2}a_1+\frac{1}{2}a_2,\ 2b+2b'+2b'';\ -,\ -,\ -,\ -, \end{matrix}\ 1,\ 1,\ 1,\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(b+b'+b''+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a_1+\frac{1}{2}a_2\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}a_1-\frac{1}{2}a_2+b+b'+b''\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a_1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a_2\right)\Gamma\left(\frac{1}{2}+b+b'+b''-2a_1\right)\Gamma\left(1+b+b'+b''-\frac{1}{2}a_2\right)}$$

(b) Whipple's theorem for three variables

$$(2.2)\quad F\left[\begin{matrix} a_1,\ 1-a_1;\ -,-,-,\ ;\ b, b', b''; \\ c_1, 1+2b+2b'+2b''-c_1;\ -,-,-,-,-, \end{matrix}\ 1,\ 1,\ 1\right]$$

$$=\frac{2^{1-2b-2b'-2b''}\ \pi\ \Gamma(c_1)\Gamma(1+2b+2b'+2b''-c_1)}{\Gamma\left(\frac{1}{2}c_1+\frac{1}{2}a_1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}c_1-\frac{1}{2}a_1\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}a_1-\frac{1}{2}c_1+b+b'+b''\right)}$$

$$\times\frac{1}{\Gamma\left(1-\frac{1}{2}a_1-\frac{1}{2}c_1+b+b'+b''\right)}$$

PROOF. Let us consider the triple series

$$(2.3)\quad F\left[\begin{matrix} (a_2);-,-,-;\ b, b', b''; \\ (c_2);-,-,-,-,-,-, \end{matrix}\ 1,\ 1,\ 1,\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{(a_1)_{m+n}(a_2)_{m+n}(b)_m(b')_n}{m!n!(c_1)_{m+n}(c_2)_{m+n}}\,{}_3F_2(a_1+m+n,\ a_2+m+n,\ b'',$$

$$c_1+m+n,\ c_2+m+n;\ 1)$$

Using the theorem (3, p. 52)

$$(2.4)\quad {}_3F_2(a, b, c, e, f, 1)=\frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\times{}_3F_2(e-a, f-a, s; s+b, s+c; 1)$$

where $s=e+f-a-b-c$; $\mathrm{Re}(s)>0$, $\mathrm{Re}(a)>0$

The Right Hand side of (2.3)

$$=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b'')\Gamma(c_1+c_2-a_1-a_2)}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(a_2)_{m+n}(b)_m(b')_n}{m!n!(c_1+c_2-a_1-b'')_{m+n}}\,{}_3F_2\left[\begin{matrix} c_1-a_1, c_2-a_1, c_1+c_2-a_1-a_2-b'' \\ c_1+c_2-a_1-b''+m+n, c_1+c_2-a_1-a_2 \end{matrix}\ 1\right]$$

$\mathrm{Re}(c_1+c_2-a_1-a_2-b'')>0$, $\mathrm{Re}(a_1)>0$.

$$=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-b'')\Gamma(c_1+c_2-a_1-a_2)}$$

$$\times\sum_{m,n,p=0}^{\infty}\frac{(a_2)_{m+n}(b)_m(b')_n(c_1-a_1)_p(c_2-a_1)_p(c_1+c_2-a_1-a_2-b'')_p}{m!n!p!\ (c_1+c_2-a_1-b'')_{m+n}(c_1+c_2-a_1-b''+m+n)_p(c_1+c_2-a_1-a_2)_p}$$

$$=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b'')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b'')}$$

$$\times\sum_{p=0}^{\infty}\frac{(c_1-a_1)_p(c_2-a_1)_p(c_1+c_2-a_1-a_2-b'')_p}{p!(c_1+c_2-a_1-a_2)_p(c_1+c_2-a_1-b'')_p}F_1(a_2;b,b';c_1+c_2-a_1-b''+p;1,1)$$

But using [1, p. 22]

$$(2.5)\quad F_1(a;b,b';c;1,1)=\frac{\Gamma(c)\Gamma(c-a-b-b')}{\Gamma(c-a)\Gamma(c-b-b')},$$

(2.3) becomes as

$$(2,6)\quad F\left[\begin{matrix}(a_2);-,-,-,;\ b,b',b'';\\(c_2);-,-,-,;-,-,-;\end{matrix}\ 1,\ 1,\ 1\right]$$

$$=\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1+c_2-a_1-a_2-b-b'-b'')}{\Gamma(a_1)\Gamma(c_1+c_2-a_1-a_2)\Gamma(c_1+c_2-a_1-b-b'-b'')}$$

$$\times{}_3F_2\left[\begin{matrix}c_1-a_1,c_2-a_1,c_1+c_2-a_1-a_2-b-b'-b'';\\c_1+c_2-a_1-a_2,c_1+c_2-a_1-b-b'-b''\end{matrix}\ 1\right]$$

Now if, we put in (2.6)

(i) $c_2=\frac{1}{2}(1+a_1+a_2)$, $c_2=2(b+b'+b'')$

and summing by Dixon's theorem [3, p. 52], we get (2.1), the Watson's theorem for three variables,

(ii) $a_2=1-a_1$ and $c_2=1+2b+2b'+2b''$ and using Watson's theorem [3, p. 54], we get (2.2) the Whipple's theorem for the variables.

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REFERENCES

[1] Appell, P. and Kampe de Feriet J., *Fonctions Hypergeometriques at Hyper spheriques*, Gauthier Villars, Paris 1926.

[2] Parihar. C. L., *A Theorem for three variables analogous to Dixon's theorem*, Univ. of Indore, Research Jour. Science. vol. 1 No. 1. 1972 pp. 14—16.

[3] Slater, L. J., *Generalized Hypergeometric Series*, Cambridge Univ. Press. London. 1966.