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ON SOME NEW RESULTS INVOLVING DOUBLE AND TRIPLE HYPERGEOMETRIC FUNCTIONS

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1. Introduction

The subject of generating relations plays an important role in the development and study of special functions. In the present paper we establish some new generating relations for certain hypergeometric functions of two and three variables. In section 4, we give some interesting particular cases.

If we use the notation $(a,n)=a(a+1)(a+1)\cdots(a+n-1)$; $(a,0)=1$, where a is arbitrary and n,a positive integer, then the hypergeometric functions of three variables F_A, F_E, F_G, G_B, ${}_3H_A^{(1)}$, ${}_3\Phi_G^{(1)}$ have been defined by Lauricella [5], Saran [8], Sharma and Mittal [9], Pandey [6], Dhawan [2], Jain [3] and Horn's functions of two variables H_1, H_5, G_1 as given in [1, p.225, (13); p.225, (17) and p.224, (10)] are respectively as follows:

$$F_A(\alpha,\beta_1,\beta_2,\beta_3;\ \gamma_1,\gamma_2,\gamma_3;x,y,z)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha,m+n+p)(\beta_1,m)(\beta_2,n)(\beta_3,p)}{(1,m)(1,n)(1,p)(\gamma_1,m)(\gamma_2,n)(\gamma_3,p)}x^m y^n z^p,\quad |x|+|y|+|z|<|; \tag{1.1}$$

$$F_E(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_2;\gamma_1,\gamma_2,\gamma_3;x,y,z)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1,m+n+p)(\beta_1,m)(\beta_2,n+p)}{(1,m)(1,n)(1,p)(\gamma_1,m)(\gamma_2,n)(\gamma_3,p)}x^m y^n z^p,\quad |x|<r,\ |y|<s,\ |z|<t,$$
$$r+(\sqrt{s}+\sqrt{t})^2=1; \tag{1.2}$$

$$F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1,m+n+p)(\beta_1,m)(\beta_2 n)(\beta_3,p)}{(1,m)(1,n)(1,p)(\gamma_1,m)(\gamma_2,n+p)}x^m y^n z^p,\quad |x|<1,|y|<1,|z|<1; \tag{1.3}$$

$$G_B(\alpha,\beta_1,\beta_2,\beta_3;\gamma;x,y,z)$$
$$=\sum_{m,n,p=0}^{\infty}\frac{(\alpha,n+p-m)(\beta_1,m)(\beta_2,n)(\beta_3,p)}{(1,m)(1,n)(1,p)(\gamma,n+p-m)}x^m y^n z^p,\quad |x|<1,\ |y|<1,|z|<1; \tag{1.4}$$

$$ {}_3H_A^{(1)}(\alpha,\beta;\gamma,\gamma';x,y,z) $$

$$ =\sum_{m,n,p=0}^{\infty}\frac{(\alpha,m+p)(\beta,m+n)}{(1,m)(1,n)(1,p)(\gamma,m)(\gamma',n+p)}x^m y^n z^p,\quad |x|<r,\quad |y|<s,\ |z|<t,\ r+s+t $$

$$ =1+st; \tag{1.5} $$

$$ {}_3\Phi_G^{(1)}(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2;\gamma_1,\gamma_2,\gamma_2;x,y,z) $$

$$ =\sum_{m,n,p=0}^{\infty}\frac{(\alpha_1,m+n+p)(\beta_1,m)(\beta_2,n)}{(1,m)(1,n)(1,p)(\gamma_1,m)(\gamma_2,n+p)}x^m y^n z^p,\quad |x|<1,\quad |y|<1; \tag{1.6} $$

$$ H_1(\alpha,\beta,\gamma,\delta,x,y) $$

$$ =\sum_{m,n=0}^{\infty}\frac{(\alpha,m-n)(\beta,m+n)(\gamma,n)}{(1,m)(1,n)(\delta,m)}x^m y^n,\quad |x|<r,\quad |y|<s,\quad 4rs=(s-1)^2; \tag{1.7} $$

$$ H_5(\alpha,\beta,\gamma,x,y) $$

$$ =\sum_{m,n=0}^{\infty}\frac{(\alpha,2m+n)(\beta,n-m)}{(1,m)(1,n)(\gamma,n)}x^m y^n,\quad |x|<r,\ |y|<s,\ 1+16r^2-36rs\pm(8r-s+27rs^2) $$

$$ =0; \tag{1.8} $$

$$ G_1(\alpha,\beta,\beta',x,y) $$

$$ =\sum_{m,n=0}^{\infty}\frac{(\alpha,m+n)(\beta,n-m)(\beta',m-n)}{(1,m)(1,n)}x^m y^n,\quad |x|<r,\ |y|<s,\ r+s=1. \tag{1.9} $$

In the present investigation we also require the following linear relations:

$$ \sum_{n=0}^{\infty}\frac{(\lambda,n)}{(1,n)}F_1(-n,\mu,\nu;\alpha;\ x,y)t^n=(1-t)^{-\lambda}F_1\Big(\lambda,\ \mu,\ \nu;\ \alpha;\frac{xt}{t-1},\ \frac{yt}{t-1}\Big), $$

$$ \max\{|xt/(t-1)|,\ |yt/(t-1)|,\ |t|\}<1; \tag{1.10} $$

$$ \sum_{n=0}^{\infty}\frac{(\lambda,n)}{(1,n)}F_2(-n,\mu,\nu;\alpha,\beta;x,y)t^n=(1-t)^{-\lambda}F_2\Big(\lambda,\mu,\nu;\alpha,\beta;\frac{xt}{t-1},\ \frac{yt}{t-1}\Big), $$

$$ \max\{|xt/(t-1)|+|yt/(t-1)|,\ |t|\}<1; \tag{1.11} $$

$$ \sum_{n=0}^{\infty}\frac{(\lambda,n)}{(1,n)}F_4(-n,\mu;\alpha,\beta;x,y)t^n=(1-t)^{-\lambda}F_4\Big(\lambda,\mu;\alpha,\beta;\frac{xt}{t-1},\ \frac{yt}{t-1}\Big), $$

$$ \max\{\sqrt{|xt/(t-1)|}+\sqrt{|yt/(t-1)|},\ |t|\}<1; \tag{1.12} $$

$$ P_n^{(\alpha,\ \beta-n)}(1-2xy)=\sum_{k=0}^{n}\frac{(\alpha+1,n)}{(1,n-k)(\alpha+1,k)}y^k(1-y)^{n-k}P_k^{(\alpha,\beta-k)}(1-2x); \tag{1.13} $$

$$ F_G(\alpha_1,\alpha_1,\alpha_1,\beta_1,\beta_2,\beta_3;\gamma_1,\gamma_2,\gamma_2;x,y,z) $$

$$=(1-x)^{-\beta_1}(1-y)^{-\beta_2}(1-z)^{-\beta_3}\, G_B\Big(1-\gamma_1, \beta_1, \beta_2, \beta_3; \gamma_2; \frac{x}{x-1}, \frac{y}{y-1}, \frac{z}{z-1}\Big),$$
$$\gamma_1+\gamma_2=1+\alpha_1; \qquad (1.14)$$

$$\Phi_2(\beta, \alpha; \gamma; x, y)=(1-x)^{-\beta}(1-y)^{-\alpha}\ \Phi_2\Big(\beta+m, \alpha+m; \gamma; \frac{x}{x-1}, \frac{y}{y-1}\Big); \qquad (1.15)$$

where F_1, F_2, F_4 are Appell's double hypergeometric functions [1, p. 224, (6); (7); (9)], Φ_2 is Horn's function [1, p. 225, (21)]. Equations (1.10), (1.11), (1.12); (1.13); (1.14) are known results given by Srivastava [10, p. 86, (4.1); (4.2); (4.3)]; [12. p. 22, (4,5)]; [11, p. 102, (1.4)] and (1.15) is due to Dhawan [2, p. 244].

2. We prove here the following formulae:

$$\sum_{n=0}^{\infty}\frac{(\lambda, n)}{(1, n)}\, H_1(\lambda+n, \beta, \gamma, \delta, x, y)\ \ F_1(-n, \mu, \nu; \alpha; u, v)\, t^n$$
$$= (1-t)^{-\lambda}\sum_{q=0}^{\infty}\frac{(\beta, q)(\gamma, q)}{(1, q)(1-\lambda, q)}\, (y(t-1))^q F_G\Big(\lambda-q, \lambda-q, \lambda-q, \beta+q, \mu, \nu; \delta, \alpha, \alpha;$$
$$\frac{x}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\Big). \qquad (2.1)$$

$$\sum_{n=0}^{\infty}\frac{(\lambda, n)}{(1, n)}\, H_5(\lambda+n, \beta, \gamma, x, y)F_4(-n, \mu; \alpha, \beta; u, v)t^n$$
$$= (1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda, 2p)}{(1, p)(1-\beta, p)}\Big(\frac{-x}{(1-t)^2}\Big)^p F_E\Big(\lambda+2p, \lambda+2p, \lambda+2p, \beta-p, \mu, \mu; \gamma, \alpha, \beta;$$
$$\frac{y}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\Big). \qquad (2.2)$$

$$\sum_{n=0}^{\infty}\frac{(\lambda, n)}{(1, n)}\, G_1(\lambda+n, \beta, \beta', x, y)F_2(-n, \mu, \nu; \alpha, \beta; u, v)t^n$$
$$= (1-t)^{-\lambda}\sum_{p=0}^{\infty}\frac{(\lambda, p)(\beta', p)}{(1, p)(1-\beta, p)}\Big(\frac{x}{t-1}\Big)^p F_A\Big(\lambda+p, \beta-p, \mu, \nu; 1-\beta'-p, \alpha, \beta;$$
$$\frac{y}{1-t}, \frac{ut}{t-1}, \frac{vt}{t-1}\Big). \qquad (2.3)$$

PROOF. To prove (2.1), consider

$$T= \sum_{n=0}^{\infty}\frac{(\lambda, n)}{(1, n)}\, H_1(\lambda+n, \beta, \gamma, \delta, x, y)F_1(-n, \mu, \nu; \alpha; u, v)t^n,$$

express H_1 in series form as given in (1.7), employ elementary relation

$$(\lambda, n)(\lambda+n, p-q)=(\lambda, n+p-q)=(\lambda, p-q)(\lambda+p-q, n), \qquad (2.4)$$

and apply relation (1.10), we find

$$T= \sum_{p,q=0}^{\infty}\frac{(\lambda, p-q)(\beta, p+q)(\gamma, q)}{(1, p)(1, q)(\delta, p)}\, F_1\Big(\lambda+p-q, \mu, \nu; \alpha;$$
$$\frac{ut}{t-1}, \frac{vt}{t-1}\Big)\, x^p\, y^q\, (1-t)^{-\lambda-p+q}. \qquad (2.5)$$

On expressing F_1 in series form and using a relation similar to (2.4), (2.5) yields

$$T=(1-t)^{-\lambda}\sum_{q=0}^{\infty}\frac{(\beta,q)(\gamma,q)}{(1,q)(1-\lambda,q)}(y(t-1))^{q}\sum_{p,r,s=0}^{\infty}\frac{(\lambda-q,p+r+s)(\beta+q,p)(\mu,r)(\nu,s)}{(1,p)(1,r)(1,s)(\delta,p)(\alpha,r+s)}$$
$$\left(\frac{x}{1-t}\right)^{p}\left(\frac{ut}{t-1}\right)^{r}\left(\frac{vt}{t-1}\right)^{s},$$

which in the light of (1.3) provides (2.1).

The proof of the formulae (2.2) and (2.3) would run parallel to what we have obtained above.

3. In this section following generating relations have been established:

$$ {}_3H_A^{(1)}\left((\alpha,\beta;\gamma,\gamma';\ x,\ \frac{y}{y-1},\ \frac{z}{z-1}\right)$$
$$=(1-y)^{\beta}(1-z)^{\alpha}\ {}_2F_1(\alpha,\beta;\gamma;x)\sum_{n=0}^{\infty}\frac{(-y)^n}{(\gamma',n)}P_n^{(-\beta-n,\alpha+\beta-1)}\left(1-\frac{2z}{y}\right) \tag{3.1}$$

$$\times(1-x)^{\alpha_1}(1-x)^{-\beta_1}(1-y(1-x))^{-\beta_2}(1-y(1-x)(1-yz))^{-\beta_3}$$
$$\times G_B\left(1-\beta_1,\beta_1,\beta_2,\beta_3;\gamma_2;\ \frac{x}{x-1},\ \frac{y(1-x)}{y(1-x)-1},\ \frac{y(1-x)(1-yz)}{y(1-x)(1-yz)-1}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(\alpha_1,k)}{(\gamma_2,k)}(yz)^k{}_2F_1(\alpha_1+k,\ \beta_2+\beta_3+k;\ \gamma_2+k;\ 1-z)P_k^{(\beta_2+\beta_3-1,-\beta_2-k)}(1-2y), \tag{3.2}$$

PROOF. To prove (3.1), let us consider

$$\Delta={}_3H_A^{(1)}\left(\alpha,\beta;\gamma,\gamma';\ x,\ \frac{y}{y-1},\ \frac{z}{z-1}\right),$$

express ${}_3H_A^{(1)}$ in series form as given in (1.5), we get

$$\Delta=\sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(1,m)(\gamma,m)}\Phi_2\left(\beta+m,\alpha+m;\gamma'\ ;\frac{y}{y-1},\ \frac{z}{z-1}\right)x^m$$

which in the light of (1.15) provides

$$\Delta=\sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(1,m)(\gamma,m)}\Phi_2(\beta,\ \alpha;\ \gamma';\ y,z)(1-y)^{\beta}(1-z)^{\alpha}\ x^m.$$

Write Φ_2 in series form and apply the series manipulation technique, we obtain

$$\Delta=(1-y)^{\beta}(1-z)^{\alpha}\sum_{m=0}^{\infty}\frac{(\alpha,m)(\beta,m)}{(1,m)(\gamma,m)}x^m\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(\alpha,\gamma)(\beta,n-r)}{(1,r)(1,n-r)(\gamma',n)}y^{n-r}z^r. \tag{3.3}$$

Now use the results [7, p. 58, (2); (3)], replace r-series in Jacobi polynomial with the help of [7, p. 254, (1)] and express m-series in ${}_2F_1$, (3.3) yields (3.1).

In order to prove (3.2), consider

$$\Omega=(1-x)^{\alpha_1}(1-x)^{-\beta_1}(1-y(1-x))^{-\beta_2}(1-y(1-x)(1-yz))^{-\beta_3}$$
$$\times G_B\left(1-\beta_1,\beta_1,\beta_2,\beta_3;\gamma_2;\frac{x}{x-1},\frac{y(1-x)}{y(1-x)-1},\frac{y(1-x)(1-yz)}{y(1-x)(1-yz)-1}\right),$$

use the relation (1.14), express F_G in series form, then on using series manipulation technique, one finds

$$\Omega=\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{(\alpha_1,n)(\beta_2,n-p)(\beta_3,p)}{(1,p)(1,n-p)(\gamma_2,n)}y^n(1-yz)^p. \tag{3.4}$$

Express p series in Jacobi polynomial and use the result

$$P_n^{(\alpha,\beta)}(-z)=(-1)^n P_n^{(\beta,\alpha)}(z),$$

(3.4) assumes the form

$$\Omega=\sum_{n=0}^{\infty}\frac{(\alpha_1,n)}{(\gamma_2,n)}P_n^{(\beta_2+\beta_3-1,-\beta_3-n)},(1-2yz)y^n \tag{3.5}$$

Finally use the relation (1.13), apply series manipulation technique and express n-series in ${}_2F_1$, (3.5) provides (3.2).

4. Particular cases

(i) In (2.1) Putting $\nu=0$ and simplifying, we get

$$\sum_{p,q=0}^{\infty}\frac{(\lambda,p-q)(\beta,p+q)(\gamma,q)}{(1,p)(1,q)(\delta,p)}\left(\frac{x}{1-t}\right)^p(y(1-t))^q\,{}_2F_1\left((\lambda+p-q,\mu;\alpha;\frac{ut}{t-1}\right)$$
$$=\sum_{q=0}^{\infty}\frac{(\beta,q)(\gamma,q)}{(1,q)(1-\lambda,q)}(y(t-1))^q F_2\left(\lambda-q,\beta+q,\mu;\delta,\alpha;\frac{x}{1-t},\frac{ut}{t-1}\right), \tag{4.1}$$

(ii) In (2.1) if we replace v by $\frac{v}{\nu}$ and take the limit as $\nu\to\infty$ and then letting $\gamma=0$, we have

$$\sum_{p=0}^{\infty}\frac{(\beta,p)(\lambda,p)}{(1,p)(\delta,p)}\left(\frac{x}{1-t}\right)^p\Phi_1\left(\lambda+p,\mu,\alpha,\frac{ut}{t-1},\frac{vt}{t-1}\right)$$
$$={}_3\Phi_G^{(1)}\left(\lambda,\lambda,\lambda,\beta,\mu;\delta,\alpha,\alpha;\frac{x}{1-t},\frac{ut}{t-1},\frac{vt}{t-1}\right), \tag{4.2}$$

where Φ_1 is Horn's function [1, p.225, (20)].

(iii) Replace z by x, express x by $\frac{x}{\alpha}$ and letting $\alpha\to\infty$, (3.1) provides

$$\Phi_3(\beta,\gamma',y,x)=\sum_{n=0}^{\infty}\frac{(-y)^n}{(\gamma',n)}L_n^{(-\beta-n)}\left(\frac{x}{y}\right), \tag{4.3}$$

which is a relation similar to given by Khan [4, p.183, (4.8)] and where Φ_3 is

Horn's function [1, p. 225, (22)].

Putting $\beta=\gamma'=-\beta$, (4.3) yields

$$\sum_{n=0}^{\infty}\frac{y^n}{(1,n)}\,{}_0F_1(-;-\beta+n;x)=\sum_{n=0}^{\infty}\frac{(1,\beta-n)}{(1,\beta)}y^n L_n^{(-\beta-n)}\left(\frac{x}{y}\right), \tag{4.4}$$

where ${}_0F_1$ represents Bessel function.

We conclude this paper with the remark that a number of results similar to given in section 4 can easily be derived from results of section 2 and 3.

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