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HARMONIC FUNCTIONS AND BOUNDARY VALUE PROBLEMS

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1. Intrdouction

A generalization of Fox's H-function [4, p. 408] has been introduced by Agarwal and Mathur [1, p. 30] and we define a generalized Fox's H-function in a slightly variant form as

$$(1.1)\qquad H\begin{bmatrix} x \\ y \end{bmatrix} \equiv H\left[\begin{array}{c|c} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} & \{(a_{p_1}, A_{p_1})\} ; \{(b_{q_1}, B_{q_1})\} \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{pmatrix} & \{(c_{p_2}, C_{p_2})\} ; \{(d_{q_2}, D_{q_2})\} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{pmatrix} & \{(e_{p_3}, E_{p_3})\} ; \{(f_{q_3}, F_{q_3})\} \end{array}\middle| \begin{array}{c} x \\ \\ y \end{array}\right]$$

$$=\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} F(\xi+\eta)G(\xi,\eta)x^{\xi}\, y^{\eta}\, d\xi\, d\eta$$

where

(i) $\{(a_{p_1}, A_{p_1})\}$ is taken to abbreviate the set of p_1-parameters:

$(a_1, A_1), (a_2, A_2), \cdots, (a_j, A_j), \cdots, (a_{p_1}, A_{p_1})$; and so on,

(ii) All A's, B's, $\cdots$, and p, m, n etc. are all positive integers satisfying $p_1\geq m_1\geq 0$, $p_2\geq m_2\geq 0$, $q_1\geq 0$, $p_3\geq m_3\geq 0$, $q_2\geq n_2\geq 0$, $q_3\geq n_3\geq 0$, $q_1+q_2\geq p_1+p_2$ and $q_1+q_3\geq p_1+p_3$;

(iii)

$$\sum_1^{p_1} A_j+\sum_1^{p_2} C_j-\sum_1^{q_1} B_j-\sum_1^{q_2} D_j<0\ ;$$

$$\sum_1^{m_1} A_j-\sum_{m_1+1}^{p_1} A_j-\sum_1^{q_1} B_j+\sum_1^{m_2} C_j-\sum_{m_2+1}^{p_2} C_j+\sum_1^{n_2} D_j-\sum_{n_2+1}^{q_2} D_j\equiv\gamma>0,\ |\arg x|<\frac{1}{2}\gamma\pi,$$

$$\sum_1^{p_1} A_j+\sum_1^{p_3} E_j-\sum_1^{q_1} B_j-\sum_1^{q_3} F_j<0\ ;$$

$$\sum_1^{m_1} A_j-\sum_{m_1+1}^{p_1} A_j-\sum_1^{q_1} B_j+\sum_1^{m_3} E_j-\sum_{m_3+1}^{p_3} E_j+\sum_1^{n_3} F_j-\sum_{n_3+1}^{q_3} F_j\equiv\nu>0,\ |\arg y|<\frac{1}{2}\nu\pi\ ;$$

$$
\text{(iv)} \begin{cases} F(\xi+\eta)=\dfrac{\prod_{j=1}^{m_1}\Gamma(1-a_j+A_j\xi+A_j\eta)}{\prod_{j=m_1+1}^{p_1}\Gamma(a_j-A_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(b_j+B_j\xi+B_j\eta)}, \\ G(\xi,\ \eta)=\dfrac{\prod_{j=1}^{m_2}\Gamma(c_j+C_j\xi)\prod_{j=1}^{n_2}\Gamma\ (d_j-D_j\xi)\prod_{j=1}^{m_3}\Gamma(e_j+E_j\eta)\prod_{j=1}^{n_3}\Gamma(f_j-F_j\eta)}{\prod_{j=m_2+1}^{p_2}\Gamma(1-c_j-C_j\xi)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+D_j\xi)\prod_{j=m_3+1}^{p_3}\Gamma(1-e_j-E_j\eta)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)}. \end{cases}
$$

The sequence of parameters $a_{m_1}, c_{m_2}, d_{n_2}, e_{m_3}$ and f_{n_3} are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(d_j-D_j\xi)$, $(1\leq j\leq n_2)$ and $\Gamma(f_j-F_j\eta)$, $(1\leq j\leq n_3)$ lie to the right and those of $\Gamma(c_j+C_j\xi)$, $(1\leq j\leq m_2)$, $\Gamma(e_j+E_j\eta)$, $(1\leq j\leq m_3)$ and $\Gamma(1-a_j+A_j\xi+A_j\eta)$, $(1\leq j\leq m_1)$ lie to the left of the imaginary axis.

The integral (1.1) is well-defined and convergent if

$$
(1.2) \begin{cases} 2(m_1+m_2+n_2)>p_1+q_1+p_2+q_2, \\ 2(m_1+m_3+n_3)>p_1+q_1+p_3+q_3, \\ |\arg x|<[m_1+m_2+n_2-\frac{1}{2}(p_1+q_1+p_2+q_2)]\pi, \\ |\arg y|<[m_1+m_3+n_3-\frac{1}{2}(p_1+q_1+p_3+q_3)]\pi. \end{cases}
$$

To economise the contents of this paper, we have used the following notations throughout:

(i) $\Delta(m,n)=\frac{n}{m},\ \frac{n+1}{m},\cdots,\ \frac{n+m-1}{m}$; m is a positive integer >0,

$$
\text{(ii)}\quad f(p,q,h)=\begin{cases} \begin{bmatrix} m_1+2h,0 \\ p_1-m_1, q_1+2h \end{bmatrix} \\ \begin{pmatrix} m_2,\ n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix} \\ \begin{pmatrix} m_3,\ n_3 \\ p_3-m_3, q_3-n_3 \end{pmatrix}, \end{cases}
$$

(iii) $\phi_1=\Delta(h,-\rho), \phi_2=\Delta(h,-\rho+\alpha),\ \theta_1=\Delta(h,\beta+l+\rho+2),\ \theta_2=\Delta(h,-\alpha-l+\rho+1)$,

(iv) Replacing ρ by $\rho+\alpha$ in (iii) to obtain $\phi_1=\phi_3,\ \phi_2=\phi_4,\ \theta_1=\varphi_1,\ \theta_2=\varphi_2$;

(v) Taking $\rho=\rho+\alpha$ and $l=k$ in (iii) to get $\phi_1=\phi_3$, $\phi_2=\phi_4$, $\theta_1=\theta_3$, $\theta_2=\theta_4$;

(vi) $\omega_1=\Delta(h, -\beta-k-\alpha-\rho-1)$, $\omega_2=\Delta(h, -\rho+k)$;

(vii) $f(l)=(-1)^l \dfrac{(2l+\sigma)\Gamma(l+\sigma)}{\Gamma(l+\alpha+1)}$; $\sigma=\alpha+\beta+1$.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $n=0,1,2,\cdots$, $\alpha>-1$ and $\beta>-1$, are generalizations of associated Legendre functions, Ultraspherical and Legendre polynomials, defined by

(a) Rodrigues' formula [5, p. 275, (7)]:

$$(1.3)\quad P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n\, n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}],$$

(b) Orthogonality-property [5, p. 276, (22)]:

$$(1.4)\qquad \int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\beta)}(x)dx=h_n\,\delta_{mn},$$

where Re $(\alpha)>-1$, Re $(\beta)>-1$,

$$h_n=\frac{2^{\sigma}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\sigma)\Gamma(n+\sigma)\; n!},\quad \sigma=\alpha+\beta+1;$$

and δ_{mn} is Kronecker's δ: $\delta_{mn}=\begin{cases}0 \text{ if } m\neq n\\ 1 \text{ if } m=n.\end{cases}$

Recently the author [6], in a previous paper, has evaluated an integral:

$$(1.5)\quad \int_{-1}^{1}(1+u)^{\beta}(1-u)^{\rho}\,P_l^{(\alpha,\beta)}(u)H\left[\begin{matrix}x\left(\frac{1-u}{2}\right)^h\\ y\left(\frac{1-u}{2}\right)^h\end{matrix}\right]du$$

$$=\frac{(-1)^l\,2^{\rho+\beta+1}\Gamma(\beta+l+1)}{l!\;h^{\beta+1}}H\left[f(p,q,h)\left|\begin{matrix}\phi_1,\phi_2,\{(a_{p_1},A_{p_1})\}:\{(b_{q_1},B_{q_1})\},\theta_1,\theta_2\\ \{(c_{p_2},C_{p_2})\}:\{(d_{q_2},D_{q_2})\}\\ \{(e_{p_3},E_{p_3})\}:\{(f_{q_3},F_{q_3})\}\end{matrix}\right|\begin{matrix}x\\ y\end{matrix}\right]$$

where, for convergence, h being a positive integer >0, Re $(\beta)>0$;

$$\text{(i)}\quad\begin{cases}p_1+q_1+p_2+q_2<2(m_1+m_2+n_2),\\ p_1+q_1+p_3+q_3<2(m_1+m_3+n_3),\\ |\arg x|<[m_1+m_2+n_2-\frac{1}{2}(p_1+q_1+p_2+q_2)]\pi,\\ |\arg y|<[m_1+m_3+n_3-\frac{1}{2}(p_1+q_1+p_3+q_3)]\pi\\ \text{Re}\left[\rho+h\frac{d_j}{D_j}+h\frac{f_i}{F_i}\right]>-1,\ (1\leq j\leq n_2;\ 1\leq i\leq n_3),\end{cases}$$

or

$$\text{(ii)} \quad \begin{cases} p_1+p_2<q_1+q_2,\ p_1+p_3<q_1+q_3, \\ \text{or else } p_1+p_2=q_1+q_2,\ p_1+p_3=q_1+q_3 \text{ with } |x|<1,\ |y|<1, \\ \mathrm{Re}\left[\rho+h\frac{d_j}{D_j}+h\frac{f_i}{F_i}\right]>-1,\ (1\leq j\leq n_2\,;\ 1\leq i\leq n_3). \end{cases}$$

In view of the known results on integral and orthogonal relation of the Jacobi polynomials, we have employed here the generalized Fox's H-functions and Jacobi's series to determine the harmonic functions $V(r,x)$ and $W(r,x)$ satisfying the partial differential equation and boundary conditions of the problems on electrostatic potential in a region of space that is free from electric charges. These results may be used in many problems of mathematics both pure and applied, and in mathematical physics appearing throughout literature. On specialization of the parameters, the solutions yield several useful results.

2. Problems on electrostatic potential in spherical regions

Here we shall utilize the generalized H-function of two variables and Jacobi's series to obtain the harmonic function V representing electrostatic potential in the domain $r<c$ such that V assumes prescribed values $F(\theta)$ on the spherical surface $r=c$ where r, φ and θ are spherical coordinates, and V is independent of φ. Thus V satisfies Laplace's equation

$$(2.1) \qquad r\frac{\partial^2}{\partial r^2}(rV)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

in the domain $r<c$, $0<\theta<\pi$ and the condition

$$(2.2) \qquad \lim_{r\to c} V=F(\theta),\ (0<\theta<\pi, r<c)\,;$$

where V and its partial derivatives of first and second order are assumed to be continuous throughout the interior $(0\leq r<c,\ 0\leq\theta\leq\pi)$ of the sphere.

Physically, the function V may represent steady temperatures in a solid sphere $r\leq c$ whose surface temperature depends on θ; that is, the surface temperature is uniform over each circle $\theta=\theta_0$, $r=c$. Here V also denotes electrostatic potential in the space $r<c$ free of charges, for $V=F(\theta)$ on the boundary $r=c$.

The equation (2.1) reduces to

$$(2.3) \qquad r\frac{\partial^2}{\partial r^2}(rV)+\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial V}{\partial x}\right]=0,\ (r<c,\ -1<x<1)$$

by introducing a new variable x, where $x=\cos\theta$ $(0\leq\theta\leq\pi)$.

If we take $F(\theta)=f(\cos\theta)=f(x)$, then $V(r, x)$ satisfies the transformed equation (2.3) and boundary conditions of the problems:

(2.4) $$\lim_{r\to c} V(r, x)=f(x),\quad (r<c,\ -1<x<1)$$

where V is continuous everywhere interior to the sphere and bounded when $0\le r\le r_0<C$.

(2.5) $$\lim_{r\to\infty} W(r, x)=0,\ (r>c)$$

where W is the harmonic function in the unbounded domain $r>c$, exterior to the spherical surface $r=c$ and rW is bounded for large values of r and for all x $(-1\le x\le 1)$.

Now we shall derive the formal solutions of our boundary value problems for

(2.6) $$f(x)=(1-x)^{\rho}\ H\left[\begin{matrix}\lambda\left(\frac{1-x}{2}\right)^{h}\\ \mu\left(\frac{1-x}{2}\right)^{h}\end{matrix}\right]$$

where h is a positive integer>0, f and f' are assumed to be sectionally continuous over the interval $(-1, 1)$.

Case I. Solution of $V(r, x)$ for $r<c$, interior to the sphere. The solution is:
(2.7) If h is a positive integer>0, $-1<x<1, r<c$, $\text{Re}(\beta)>-1$ then

$$V(r, x)=\frac{2^{\cdot}}{h^{\beta+1}}\sum_{k=0}^{\infty} f(k)\left(\frac{r}{c}\right)^{k}$$

$$\times H\left[f(p, q, h)\left|\begin{matrix}\phi_3, \phi_4, \{(a_{p_1}, A_{p_1})\} : \{(b_{q_2}, B_{q_2})\}, \theta_3, \theta_4\\ \{(c_{p_2}, C_{p_2})\} : \{(d_{q_2}, D_{q_2})\}\\ \{(e_{p_3}, E_{p_3})\} : \{(f_{q_3}, F_{q_3})\}\end{matrix}\right|\begin{matrix}\lambda\\ \\ \mu\end{matrix}\right] P_k^{(\alpha,\beta)}(x)$$

provided

(i) $$\begin{cases} p_1+q_1+p_2+q_2<2(m_1+m_2+n_2),\\ p_1+q_1+p_3+q_3<2(m_1+m_3+n_3),\\ |\arg\lambda|<\left[m_1+m_2+n_2-\frac{1}{2}(p_1+q_1+p_2+q_2)\right]\pi\\ |\arg\mu|<\left[m_1+m_3+n_3-\frac{1}{2}(p_1+q_1+p_3+q_3)\right]\pi,\\ \text{Re}\left[\rho+\alpha+h\frac{d_j}{D_j}+h\frac{f_i}{F_i}\right]>-1, (1\le j\le n_2 ; 1\le i\le n_3),\end{cases}$$

$$\text{(ii)} \begin{cases} p_1+p_2<q_1+q_2,\ p_1+p_3<q_1+q_3, \\ \text{or else } p_1+p_2=q_1+q_2,\ p_1+p_3=q_1+q_3 \text{ with } |\lambda|<1,\ |\mu|<1, \\ \mathrm{Re}\left[\rho+\alpha+h\frac{d_j}{D_j}+h\frac{f_j}{F_j}\right]>-1, (1\le j\le n_2\ ;\ 1\le i\le n_3). \end{cases}$$

PROOF. On using [3, p. 218, (8)] :

$$V(r,x)=\sum_{k=0}^{\infty} M_k\left(\frac{r}{c}\right)^k P_k^{(\alpha,\beta)}(x),\ (r\le c) \tag{2.8}$$

the solution of our boundary value problem, we find that the coefficients M_k are such that $V(c,x)=f(x)$: that is,

$$f(x)=(1-x)^{\rho}H\begin{pmatrix} \lambda\left(\frac{1-x}{2}\right)^h \\ \mu\left(\frac{1-x}{2}\right)^h \end{pmatrix}=\sum_{k=0}^{\infty} M_k\ P_k^{(\alpha,\beta)}(x), \tag{2.9}$$

where $f(x)$ is continuous in the closed interval $-1\le x\le 1$ and has a piecewise continuous derivative there, then with $\alpha>-1$, $\beta>-1$, the Jacobi series in (2.9) associated with $f(x)$ converges uniformly to $f(x)$ in $-1+\varepsilon\le x\le 1-\varepsilon, 0<\varepsilon<1$.

From (2.9) we obtain M_k in a purely formal manner. Therefore, in view of (2.9) it follows that

$$\int_{-1}^{1}(1-x)^{\rho+\alpha}(1+x)^{\beta}P_l^{(\alpha,\beta)}(x)\ H\begin{bmatrix} \lambda\left(\frac{1-x}{2}\right)^h \\ \mu\left(\frac{1-x}{2}\right)^h \end{bmatrix}dx \tag{2.10}$$

$$=\sum_{k=0}^{\infty} M_k\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_l^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(x)dx.$$

We employ (1.4) and (1.5) in (2.10) to obtain

$$M_l=\frac{2^{\rho}f(l)}{h^{\beta+1}}H\left[f(p,q,h)\left|\begin{matrix} \phi_3, \phi_4, \{(a_{p_1}, A_{p_1})\}\ :\ \{(b_{q_1}, B_{q_1})\}, \varphi_1, \varphi_2 \\ \{(c_{p_2}, C_{p_2})\}\ ;\ \{(d_{q_2}, D_{q_2})\} \\ \{(e_{p_3}, E_{p_3})\}\ ;\ \{(f_{q_3}, F_{q_3})\} \end{matrix}\right|\begin{matrix}\lambda \\ \\ \mu\end{matrix}\right]. \tag{2.11}$$

Finally, the use of (2.11) in (2.8) yields the desired result. This completes the proof of the solution (2.7).

Case II. Solution of $W(r,x)$ for $r>c$, exterior to the spherical surface.

We begin by considering the soution [3, p. 219, (12)] :

$$(2.12) \qquad W(r,x)=\sum_{k=0}^{\infty} M_k\left(\frac{c}{r}\right)^{k+1} P_k^{(\alpha,\beta)}(x), \quad (r\geq c).$$

In the formula for W, if we replace c/r by s/c, we find that rW is bounded for large values of r ($s\leq c_0\leq c$) and for all x ($-1\leq x\leq 1$). Therefore, with the aid of (2.6), we have

$$(2.13) \qquad f(x)=(1-x)^{\rho} H\left[\begin{matrix}\lambda\left(\frac{1-x}{2}\right)^h \\ \mu\left(\frac{1-x}{2}\right)^h\end{matrix}\right]=\sum_{k=0}^{\infty} M_k P_k^{(\alpha,\beta)}(x)$$

where M_k have the values from (2.11).

The solution (2.12) for the harmonic function W in the external region can be established in the same manner. Thus we obtain

$$(2.14) \quad W(r,x)=\frac{2^{\rho}}{h^{\beta+1}}\sum_{k=0}^{\infty} f(k)\,(c/r)^{k+1}$$

$$\times H\left[f(p,q,h)\left|\begin{matrix}\phi_3,\phi_4,\{(a_{p_1},A_{p_1})\}:\{(b_{q_1},B_{q_1})\},\theta_3,\theta_4\\ \{(c_{p_2},C_{p_2})\}:\{(d_{q_2},D_{q_2})\}\\ \{(e_{p_3},E_{p_3})\}:\{(f_{q_3},F_{q_3})\}\end{matrix}\right|\begin{matrix}\lambda\\ \mu\end{matrix}\right] P_k^{(\alpha,\beta)}(x)$$

valid under the conditions enumerated below:

h is a positive integer >0, $-1<x<1$, $r\geq c$, $\mathrm{Re}(\beta)>-1$,

$$\text{(i)}\left\{\begin{array}{l} p_1+q_1+p_2+q_2<2(m_1+m_2+n_2),\\ p_1+q_1+p_3+q_3<2(m_1+m_3+n_3),\\ |\arg\lambda|<\left[m_1+m_2+n_2-\frac{1}{2}(p_1+q_1+p_2+q_2)\right]\pi,\\ |\arg\mu|<\left[m_1+m_3+n_3-\frac{1}{2}(p_1+q_1+p_3+q_3)\right]\pi\\ \mathrm{Re}\left[\rho+\alpha+h\frac{d_j}{D_j}+h\frac{f_i}{F_i}\right]>-1,\quad (1\leq j\leq n_2\,;\,1\leq i\leq n_3),\end{array}\right.$$

or

$$\text{(ii)}\left\{\begin{array}{l} p_1+p_2<q_1+q_2,\ p_1+p_3<q_1+q_3,\\ \text{or else } p_1+p_2=q_1+q_2,\ p_1+p_3=q_2+q_3 \text{ with } |\lambda|<1,\ |\mu|<1,\\ \mathrm{Re}\left[\rho+\alpha+h\frac{d_j}{D_j}+h\frac{f_i}{F_i}\right]>-1,\ (1\leq j\leq n_2\,;\,1\leq i\leq n_3).\end{array}\right.$$

3. Particular cases

(a) By appropriate use of parameters in (2.7) and (2.14) and employ $A_j=B_i=\cdots$etc., $=1(1\leq j\leq p_1, 1\leq i\leq q_1, \cdots$etc.), we get the solutions involving the generalized Meijer's G-functions of two variables [2, p.537] which include many important functions of mathematical physics.

(b) In (2.7) and (2.14), if we set $p_1=p_3=m_1=m_3=n_3-1=q_3-1=0$ and replace $p_1+m_2, p_1+p_2, q_1+q_2$ and n_2 by m, p, q and n respectively along with proper choice of parameters etc., and then let $y\to 0$, we obtain

$$(3.1)\quad V(r,x)=\frac{2^{\rho}}{h^{\beta+1}}\sum_{k=0}^{\infty} f(k)\left(\frac{r}{c}\right)^{k} H^{n,m+2h}_{p+2h,q+2h}\left[x\left|\begin{matrix}\phi_3,\phi_4,\{(c_p,C_p)\}\\ \{(d_q,D_q)\},\omega_1,\omega_2\end{matrix}\right.\right]P_k^{(\alpha,\beta)}(x),\ (r\leq c)$$

and

$$(3.2)\quad W(r,x)=\frac{2^{\rho}}{h^{\beta+1}}\sum_{k=0}^{\infty} f(k)\left(\frac{c}{r}\right)^{k+1} H^{n,m+2h}_{p+2h,q+2h}\left[x\left|\begin{matrix}\phi_3,\phi_4,\{(c_p,C_p)\}\\ \{(d_q,D_q)\},\omega_1,\omega_2\end{matrix}\right.\right]P_k^{(\alpha,\beta)}(x),\ (r\geq c)$$

where the conditions of validity for these situations are described by (2.7) and (2.14) with necessary changes.

When the appropriate substitutions are made in (2.7) and (2.14), the H-functions of two variables not only include Fox's H- and Meijer's G-functions of single variable as particular cases but also incorporate most of the frequently used special functions in two arguments, i.e. Kampé de Fériet's double hypergeometric functions which, in turn yield Appell's functions which, in turn yield Appell's functions and the Whittaker functions of two variables, etc.

Therefore, the solutions appeared in this paper are of general character and encompass numerous interesting results which are oftenly required in the mathematical analysis.

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