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AN INTEGRAL INVOLVING JACOBI POLYNOMIAL AND THE H-FUNCTION

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1. Introduction

In this paper, an integral involving product of the Jacobi polynomial and the H-function has been evaluated. From this result, many other integrals involving these functions have been deduced. A large number of special functions; for instance, G-function, generalized hypergeometric functions considered by Fox and Wright, generalized Bessel function due to E. M. Wright, functions of G. Mittag-Leffler and Boersma; are particular cases of the H-function. So on specializing the parameters of the H-function involved in the integrals, many new as well as known integrals can be obtained as particular cases.

The H-function was introduced by Fox [5, p. 408] and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [2]. Following the definition given by Braaksma [2, pp. 239—241], it will be represented as follows:

$$(1.1)\quad H^{p,q}_{r,s}\left[z\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{p}\Gamma(b_j-\beta_j\zeta)\prod_{j=1}^{q}\Gamma(1-a_j+\alpha_j\zeta)z^{\zeta}}{\prod_{j=p+1}^{s}\Gamma(1-b_j+B_j\zeta)\prod_{j=q+1}^{r}\Gamma(a_j-\alpha_j\zeta)}d\zeta,$$

where $\{(a_r,\alpha_r)\}$, represents the set of parameters $(a_1,\alpha_1),\cdots,(a_r,\alpha_r)$.

In what follows, for the sake of brevity

$$\sum_1^s(\beta_j)-\sum_1^r(\alpha_j)\equiv A,\ \sum_1^p(\beta_j)-\sum_{p+1}^s(\beta_j)+\sum_1^q(\alpha_j)-\sum_{q+1}^r(\alpha_j)\equiv B,$$
$$\alpha(\alpha+1)\cdots\cdots(\alpha+n-1)=(\alpha)_n,\ n\geq 1,(\alpha)_0=1,\alpha\neq 0.$$

2. The integral to be established is

$$(2.1)\qquad \int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)H^{p,q}_{r,s}\left[z(1-x)^{\delta}(1+x)^{\mu}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N}{N!\Gamma(1+\alpha+N)}$$

$$\times H_{r+2,s+1}^{p,q+2}\left[2^{\delta+\mu}z\,\middle|\,\begin{matrix}(-\sigma,\mu),(-\rho-N,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,\delta+\mu),\end{matrix}\right],$$

where n is a positive integer, $A\geq 0$, $\mathrm{Re}(\beta)<1$, $\mathrm{Re}\ (\rho+\delta b_j/\beta_j)>-1$, $\mathrm{Re}\ (\sigma+\mu b_j/\beta_j)>-1 (j=1,2,\cdots,p)$, $B>0$, $|\arg z|<\frac{1}{2}B\pi$ and $\mu\geq 0$, $\delta>0$ (or $\delta\geq 0,\mu>0$).

PROOF. To establish (2.1), expressing the H-function as Mellin-Barnes type of contour integral (1.1), interchanging the order of integration which is justifiable due to the absolute convergence of integral involved in the process, we get

(2.2)
$$\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{p}\Gamma(b_j-\beta_j\zeta)\prod_{j=1}^{q}\Gamma(1-a_j+\alpha_j\zeta)}{\prod_{j=p+1}^{s}(1-b_j+\beta_j\zeta)\prod_{j=q+1}^{r}\Gamma(a_j-\alpha_j\zeta)}z^{\zeta}$$

$$\times\int_{-1}^{1}(1-x)^{\rho+\delta\zeta}(1+x)^{\sigma+\mu\zeta}P_n^{(\alpha,\beta)}(x)dxd\zeta.$$

Evaluating the inner integral with the help of [4, p. 284(3)]*, i.e.

(2.3)
$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)dx$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(\rho+1)\Gamma(\sigma+1)\Gamma(n+\alpha+1)}{n!\Gamma(\rho+\sigma+2)\Gamma(\alpha+1)}{}_3F_2\left[\begin{matrix}-n,\alpha+\beta+n+1,\rho+1\ ;\ 1\\ \alpha+1,\rho+\sigma+2,\end{matrix}\right],$$

provided $\mathrm{Re}(\rho)>-1$, $\mathrm{Re}(\sigma)>-1$; (2.2) reduces to:

(2.4)
$$\frac{1}{2\pi i}\int_T\frac{\prod_{j=1}^{p}\Gamma(b_j-\beta_j\zeta)\prod_{j=1}^{q}\Gamma(1-a_j+\alpha_j\zeta)\Gamma(\rho+1+\delta\zeta)\Gamma(\sigma+1+\mu\zeta)}{\prod_{j=p+1}^{s}\Gamma(1-b_j+\beta_j\zeta)\prod_{j=q+1}^{r}(a_j-\alpha_j\zeta)\Gamma(\alpha+1)n!}$$

$$\times\frac{2^{\rho+\sigma+1+(\delta+\mu)\zeta}z^{\zeta}}{\Gamma(\rho+\sigma+2+(\delta+\mu)\zeta)}{}_3F_2\left[\begin{matrix}-n,\alpha+\beta+n+1,\rho+1+\delta\zeta\ ;\ 1\\ \alpha+1,\rho+\sigma+2+(\delta+\mu)\zeta\end{matrix}\right]d\zeta.$$

Now, expressing the hypergeometric function as series, changing the order of summation and integration in view of [3, p. 176(75)], which is permissible under the conditions given in (2.3) and (2.1) and applying (1.1), the definition of the H-function, the value of the integral is obtained.

* In the result [4, p. 284(3)], the factor $\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)}$ is missing on the right-hand side.

3. In this section, we discuss some interesting particular cases of the result (2.1).

(i) Taking $\delta=0$ in (2.1), we get

$$(3.1)\qquad \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_{n}^{(\alpha,\beta)}(x) H_{r,s}^{p,q}\left[z(1+x)^{\mu}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N\Gamma(1+\rho+N)}{N!\,\Gamma(\alpha+1+N)}$$

$$\times H_{r+1,s+1}^{p,q+1}\left[2^{\mu}z\left|\begin{matrix}(-\sigma,\mu),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,\mu)\end{matrix}\right.\right],$$

where n is a positive integer, $\mu>0$, Re $(\beta)<1$, Re $(\rho)>-1$, Re $(\sigma+\mu b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq 0$, $B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

(ii) In (3.1), replacing x by $1-2x$ (or x by $2x-1$) and $2^{\mu}z$ by z, we have

$$(3.2)\qquad \int_{0}^{1} x^{\rho}(1-x)^{\sigma} P_{n}^{(\alpha,\beta)}(1-2x) H_{r,s}^{p,q}\left[z(1-x)^{\mu}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\int_{0}^{1} x^{\sigma}(1-x)^{\rho} P_{n}^{(\alpha,\beta)}(2x-1) H_{r,s}^{p,q}\left[zx^{\mu}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N\Gamma(1+\rho+N)}{N!\,\Gamma(\alpha+1+N)}$$

$$\times H_{r+1,s+1}^{p,q+1}\left[z\left|\begin{matrix}(-\sigma,\mu),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,\mu)\end{matrix}\right.\right],$$

where n is a positive integer, $\mu>0$, Re $(\beta)<1$, Re $(\rho)>-1$, Re $(\sigma+\mu b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq 0$, $B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

(iii) In (3.1), setting $\rho=\alpha$ and on the righthand side expressing the H-function as Mellin-Barnes type of integral, interchanging the order of integration and summation, evaluating the series inside the integral with the help of Gauss' theorem [6, p. 144] and again using (1.1), the definition of the H-function, we get a result established earlier by the author [1, (3.4)].

(iv) Putting $\mu=0$ in (2.1), we have

$$(3.3)\qquad \int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma} P_{n}^{(\alpha,\beta)}(x) H_{r,s}^{p,q}\left[z(1-x)^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(\sigma+1)\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N}{N!\Gamma(\alpha+1+N)}$$

$$\times H_{r+1,s+1}^{p,q+1}\left[2^{\delta}z\left|\begin{matrix}(-\rho-N,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,\delta)\end{matrix}\right.\right],$$

where n is a positive integer, $\delta>0$, Re $(\beta)<1$, Re $(\sigma)>-1$, Re $(\rho+\delta b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq0, B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

(v) In (3.3), replacing x by $1-2x$ (or x by $2x-1$) and $2^{\delta}z$ by z, we get

$$\int_0^1 x^{\rho}(1-x)^{\sigma}P_n^{(\alpha,\beta)}(1-2x)H_{r,s}^{p,q}\left[zx^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx \tag{3.4}$$

$$=\int_0^1 (1-x)^{\rho}x^{\sigma}P_n^{(\alpha,\beta)}(2x-1)H_{r,s}^{p,q}\left[z(1-x)^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{\Gamma(\sigma+1)\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N}{N!\Gamma(\alpha+1+N)}$$

$$\times H_{r+1,s+1}^{p,q+1}\left[z\left|\begin{matrix}(-\rho-N,),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,\delta)\end{matrix}\right.\right],$$

where n is a positive integer, $\delta>0$, Re $(\beta)<1$, Re $(\sigma)>-1$, Re $(\rho+\delta b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq0, B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

(vi) In (3.3), putting $\sigma=\beta$ and on the right-hand side substituting from (1.1), interchanging the order of summation and integration, summing the inner series with the help of [7, p. 49], i.e.

$${}_3F_2\left[\begin{matrix}\alpha,\beta,-n\,;1\\ \gamma,1+\alpha+\beta-\gamma-n\end{matrix}\right]=\frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma)_n(\gamma-\alpha-\beta)_n} \tag{3.5}$$

and then using [7, p. 18(9iii)], i.e.

$$\frac{\Gamma(n+1)}{\Gamma(1+n-s)}=(-1)^s(-n)_s \tag{3.6}$$

and (1.1), the definition of the H-function, we obtain another result due to author [1, (3.2)].

(vii) Taking $\mu=\delta$ in (2.1), we get

$$\int_{-1}^{1}(1-x)^{\rho}(1+x)^{\sigma}P_n^{(\alpha,\beta)}(x)H_{r,s}^{p,q}\left[z(1-x^2)^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx \tag{3.7}$$

$$=\frac{2^{\rho+\sigma+1}\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N}{N!\Gamma(\alpha+1+N)}$$

$$\times H_{r+2,s+1}^{p,q+2}\left[2^{2\delta}z\left|\begin{matrix}(-\sigma,\delta),(-\rho-N,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-1-\rho-\sigma-N,2\delta)\end{matrix}\right.\right],$$

where n is a positive integer, $\delta>0$, Re $(\beta)<1$, Re $(\rho+\delta b_j/\beta_j)>-1$, Re $(\sigma+\delta b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq0, B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

(viii) In (3.7), replacing x by $1-2x$ (or x by $2x-1$) and $2^{2\delta}z$ by z, we get:

(3.8) $$\int_0^1 x^{\rho}(1-x)^{\sigma}P_n^{(\alpha,\beta)}(1-2x)H_{r,s}^{p,q}\left[zx^{\delta}(1-x)^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\int_0^1 x^{\sigma}(1-x)^{\rho}P_n^{(\alpha,\beta)}(2x-1)H_{r,s}^{p,q}\left[zx^{\delta}(1-x)^{\delta}\left|\begin{matrix}\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right.\right]dx$$

$$=\frac{\Gamma(n+\alpha+1)}{n!}\sum_{N=0}^{n}\frac{(-n)_N(\alpha+\beta+n+1)_N}{N!\Gamma(\alpha+1+N)}$$

$$\times H_{r+2,s+1}^{p,q+2}\left[z\left|\begin{matrix}(-\sigma,\delta),(-\rho-N,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(-\rho-\sigma-N-1,2\delta)\end{matrix}\right.\right],$$

where n is a positive integer, $\delta>0$, Re $(\beta)<1$, Re $(\rho+\delta b_j/\beta_j)>-1$, Re $(\sigma+\delta b_j/\beta_j)>-1$ $(j=1,2,\cdots,p)$, $A\geq0, B>0$ and $|\arg z|<\frac{1}{2}B\pi$.

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