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NON-HOMOGENEOUS GENERALIZED HYPERGEOMETRIC TRANSFORM

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1. Introduction

The non-homogeneous hypergeometric function has been studied by Babister (1-p.265) and is defined as below:

$$ {}_pB_q[\alpha_r ; \beta_t ; z] = {}_pB_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q\end{matrix} ; z\right] $$

$$ = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \Omega(\alpha_{p+n}, \beta_{q+n} ; 0) \frac{z^n}{n!} \tag{1.1} $$

where

$$ (\beta_\nu)_n = \frac{\Gamma(\beta_\nu + n)}{\Gamma(\beta_\nu)}, \ (n=1,2,3,\cdots), \ (\alpha_\nu)_n = \frac{\Gamma(\alpha_\nu + n)}{\Gamma(\alpha_\nu)}, \ (n=1,2 \cdots), $$

and Ω is the generalized modified struve function defined by

$$ \Omega(a : c : z) = i(2\pi)^{-1} e^{+i\pi a} \Gamma(1-a) \Gamma(1+a-c) \omega, $$

where

$$ \omega = \frac{i2^{-c}}{\pi} e^{-ia\pi(a+c)} e^{\frac{1}{2}z} \Big[(1-e^{2\pi ia}) \int_0^{(1+)} e^{\frac{1}{2}zu} (1+u)^{a-1} (1-u)^{c-a-1} du $$

$$ + \{1-e^{2\pi i(c-a)}\} \int_0^{(-1+)} e^{\frac{1}{2}zv} (1+u)^{a-1} (1-u)^{c-a-1} du \Big] \tag{1.2} $$

if Re $\beta_q >$ Re α_p, the series in (1.1) converges for all z, if $p \leq q$, converges for $|z| < 1$, if $p = q+1$ and diverges for all non-zero z if $p = q+1$. We shall confine ourselves to the case $2 \leq p \leq q+1$. The function satisfies a non-homogeneous hypergeometric differential equation. If α_p is zero or a negative integer

$$ {}_pB_q[\alpha_r ; \beta_t ; z] = -{}_pF_q[\alpha_r ; \beta_t ; z] $$

and if $\beta_q - \alpha_p$ is zero or a non-negative integer,

$$ {}_pB_q[\alpha_r ; \beta_t ; z] = {}_pF_q[\alpha_r ; \beta_t ; z] $$

when any $\alpha_m (m=1, \cdots, p-1)$ is zero,

$${}_pB_q[\alpha_r ; \beta_t ; z] = \Omega(\alpha_p, \beta_q ; z),$$

To begin with we note the following integral

$$\int_0^\infty e^{-az} {}_pB_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q\end{matrix} : (bz)^k\right] z^{\mu-1} dz$$

$$= \Gamma(\mu) a^{-\mu} {}_{p+\kappa}B_q\left[\begin{matrix}\frac{\mu}{\kappa}, \frac{\mu+1}{\kappa}, \cdots, \frac{\mu+\kappa-1}{\kappa} ; \alpha_1, \cdots, \alpha_p ; \\ \beta_1 \cdots\cdots\cdots\cdots \beta_q \qquad ;\end{matrix} \left(\frac{\kappa b}{a}\right)^\kappa\right] \quad (1.3)$$

where Re $a \geq 0$ if $p+\kappa < q+1$, Re $a > |\text{Re } (\kappa b)|$ if $p+\kappa = q+1$.

In this paper we shall study the transformation

$$\Phi(p) = \int_0^\infty \exp(-cpx) \; {}_pB_q\left(\begin{matrix}\alpha_1, \cdots, \alpha_p \\ \beta_1, \cdots, \beta_q\end{matrix} ; px\right) f(x) dx \quad (1.4)$$

2. Inversion theorem

If $\Phi(p)$ is the non-homogeneous generalized hypergeometric transform, viz

$$\Phi(p) = \int_0^\infty K(px) f(x) dx \quad (2.1)$$

where

$$K(px) = \exp(-cpx)\, {}_lB_m\left(\begin{matrix}\alpha_l \\ \beta_m\end{matrix} ; px\right) \quad (2.2)$$

then

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{\sigma - i\tau}^{\sigma + i\tau} \Psi(s) M(s) ds \quad (2.3)$$

where

$$\Psi(s) = \frac{1}{\Gamma(1-s)} \; c^{1-s}\left[{}_{l+1}B_m\left(\begin{matrix}1-s, \alpha_1, \cdots, \alpha_l \\ \beta_1, \cdots\cdots, \beta_m\end{matrix} ; \frac{1}{c}\right)\right]^{-1} \quad (2.4)$$

and

$$M(s) = \int_0^\infty \phi^{-s} \phi(p) dp \quad (2.5)$$

provided that

(i) *the integrals* $\int_0^\infty t^{\sigma-1} f(t) dt$ *and* $\int_0^\infty t^{-s} \varphi(t) dt$, *are absolutely convergent*

$(s=\sigma+i\tau,\ 0<\tau<\infty)$

(ii) *$f(t)$ is of bounded variation in the neighbourhood of the point $t=x$.*

(iii) *$f(t)=0(t^{\lambda})$ for small t,* Re $(1+\lambda)>0$ Re $c\geq 0$ *if $l<m$,* Re $c>1$ *if $l=m$.*

(iv) *$K(p_0 t)f(t)$ is bounded for $t\geq 0$.*

PROOF. Multiplying both sides of (2.1) by p^{-s} and integrating with respect to p from 0 to ∞, we get

$$M(s)=\int_0^\infty p^{-s}\,dp\int_0^\infty \exp(-cpx)\ {}_lB_m\left(\begin{matrix}\alpha_l\\ \beta_m\end{matrix};px\right) f(x)\,dx \tag{2.6}$$

$$=\int_0^\infty x^{s-1}f(x)dx\int_0^\infty (px)^{-s}\exp(-cpx)\,{}_lB_m\left(\begin{matrix}\alpha_l\\ \beta_m\end{matrix};px\right)d(px),$$

on changing the order of integration which will be shown to be justified. Evaluating the second integral on the right by means of the formula (1.3) we obtain the following value of $M(s)$.

$$M(s)=\mu(1-s)e^{-1+s}\,{}_{l+1}B_m\left(\begin{matrix}1-s,\ \alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};\frac{1}{c}\right)\int_0^\infty x^{s-1}f(x)dx \tag{2.7}$$

where Re $c\geq 0$ if $l<m$ and Re $c>1$ if $l=m$.

Thus

$$\int_0^\infty x^{s-1}f(x)\,dx=M(s)\,\Psi(s), \tag{2.8}$$

where $\Psi(s)$ and $M(s)$ have the values given by (2.4) and (2.5) respectively. Applying Mellin's inversion formula to (2.7) we get (2.3) provided that the conditions (i) and (ii) are satisfied.

Now, since the modulus of the integrand in (2.6) is

$$p^{-\sigma}|e^{-cpx}|\,|{}_pB_q\left(\begin{matrix}\alpha_p\\ \beta_q\end{matrix};px\right)|\,|f(x)|$$

and from (1.1) ${}_pB_q(a\,;c\,;x)=o(1)$ for small values of x, it follows that the right hand side of (2.6) is absolutely convergent under the conditions (iii) and (iv) of the theorem, consequently we may change the order of integration and integrate first with respect to x.

COROLLARY 1. *If α_l be zero or a negative integer then*

$$\Phi(p)=-\int_0^\infty \exp(-cpx)\ {}_lF_m\left(\begin{matrix}\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};px\right)f(x)dx.$$

The inversion formula is

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\Psi(s)M(s)x^{-s}\,ds$$

where $\Psi(s)$ *is the same as given* (2.4) *in which* α_l *is zero or a negative integer.*

COROLLARY 2. *If* $\beta_m-\alpha_l$ *is zero or a negative integer, then*

$$\Phi(p)=\int_0^\infty \exp(-cpx)_lF_m\left(\begin{matrix}\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};px\right)f(x)\ dx$$

the inversion formula is

$$\frac{1}{2}[f(x+)+f(x-)]=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\Psi(s)M(s)x^{-s}\,ds$$

where $\Psi(s)$ *is the same as defined in* (2.4) *in which* α_l *is zero or a negative integer.*

COROLLARY 3. *If any* α_m $(m=1,\cdots,b-1)$ *is zero then* (2.1) *reduces to the Laplace transform*

$$\Phi(p)=\int_0^\infty \exp(-cpx)\Omega(\alpha_l,\beta_m;0)\,f(x)dx$$

and the corresponding formula is

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi i}\lim_{\tau\leftarrow\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\Psi(s)M(s)x^{-s}\,ds$$

where

$$\Psi(s)=\frac{1}{\Gamma(1-s)}\ c^{-s+1}[\Omega(\alpha_l,\beta_m;0)]$$

which is well known classical result on Laplace transform.

EXAMPLE. Let $f(x)=e^{-bx}$ then

$$\Phi(p)=\int_0^\infty e^{-(b+cp)x}\,{}_lB_m\left(\begin{matrix}\alpha_l,\\ \beta_m,\end{matrix}\ px\right)dx$$

$$=(b+cp)^{-1}{}_{l+1}B\left[\begin{matrix}1,\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};\frac{p}{b+cp}\right]$$

where

$$\operatorname{Re}\,(b+cp)\geq 0 \ \text{if} \ l<m, \quad \operatorname{Re}\,(b+cp)>\operatorname{Re}\,p \ \ \text{if} \ l=m$$

so that

$$M(s)=\int_0^\infty p^{-s}(b+cp)^{-1}\,{}_{l+1}B_m\left[\begin{matrix}1,\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix}:\frac{p}{b+cp}\right]dp$$

$$=\sum_{n=0}^\infty \frac{\omega_n(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n n!}\,\Omega(\alpha_{l+m},\beta_{m+n}\,;0)\int_0^\infty \frac{p^{n-s}}{(b+cp)^{n+1}}dp$$

$$=\sum_{n=0}^\infty \frac{(1)_n\,(\alpha_1)_n\cdots(\alpha_l)_n}{(\beta_1)_n\cdots(\beta_m)_n n!}\Omega(\alpha_{p+n},\beta_{q+n}\,:\,0)\frac{1}{b^{n+1}}\int_0^\infty \frac{p^{n-s+1-1}}{\left(1+\frac{c}{b}\right)^{n+1}}dp$$

$$=\frac{c^{s-1}}{b^s}\Gamma(s)\Gamma(1+s)\sum_{n=0}^\infty \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}\,\Omega(\alpha_{p+n},\beta_{q+n}\,;0)c^{-n}$$

$$=\Gamma(s)\Gamma(1-s)\frac{c^{s-1}}{b^s}\,{}_{l+2}B_{m+1}\left(\begin{matrix}1,\alpha_1,\cdots,\alpha_l,1-s\\ \beta_1,\cdots,\beta_m\end{matrix}\;;\frac{1}{c}\right),$$

where $0<\operatorname{Re}s<1$.

Hence

$$\frac{1}{2}[f(x+)+f(x-)]$$

$$=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\frac{\Gamma(s)\,b^{-s}}{{}_{l+1}B_m\left(\begin{matrix}1-s,\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};\frac{1}{c}\right)}$$

$$\times\,{}_{l+2}B_{m+1}\left(\begin{matrix}\rho,\alpha_1,\cdots,\alpha_l,1-s\\ \beta_1,\cdots,\beta_m\end{matrix}\;;\frac{1}{c}\right)x^{-s}ds$$

$$=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\Gamma(s)\frac{{}_{l+2}B_{m+1}\left(\begin{matrix}\alpha_1,\cdots,\alpha_l,1-s\\ \beta_1,\cdots,\beta_m\end{matrix};\frac{1}{c}\right)x^{-s}\left(\frac{1}{b}\right)^s ds}{{}_{l+1}B_m\left(\begin{matrix}1-s,\alpha_1,\cdots,\alpha_l\\ \beta_1,\cdots,\beta_m\end{matrix};\frac{1}{c}\right)}$$

$$=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\Gamma(s)(bx)^s ds$$

$$=e^{-bs}.$$

Thus the inversion theorm is verified.

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